A journey into the turbulent velocity gradient dynamics

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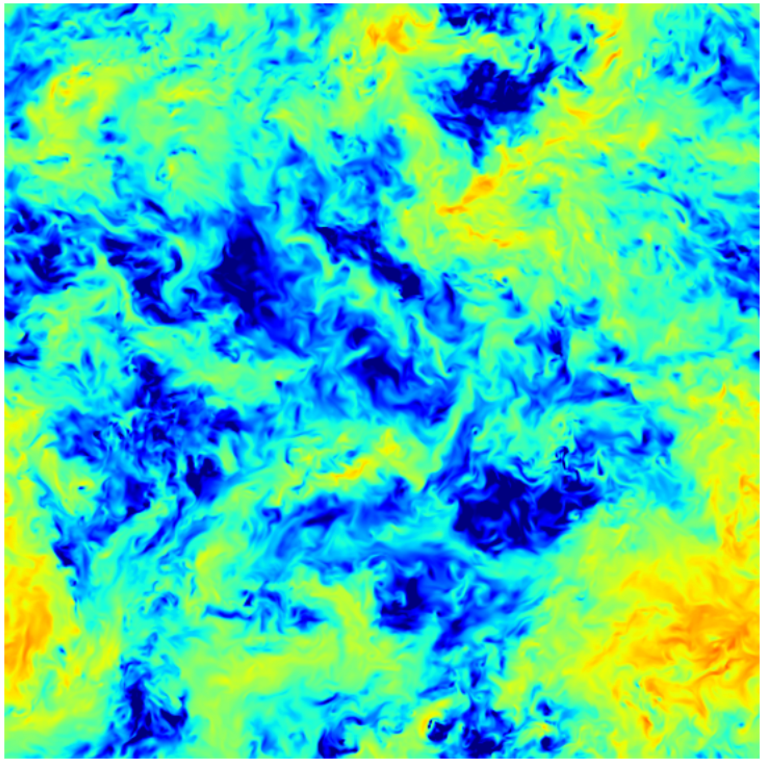
Università degli Studi di Roma "Tor Vergata"
October 3, 2023

Multi-scale nature of turbulence

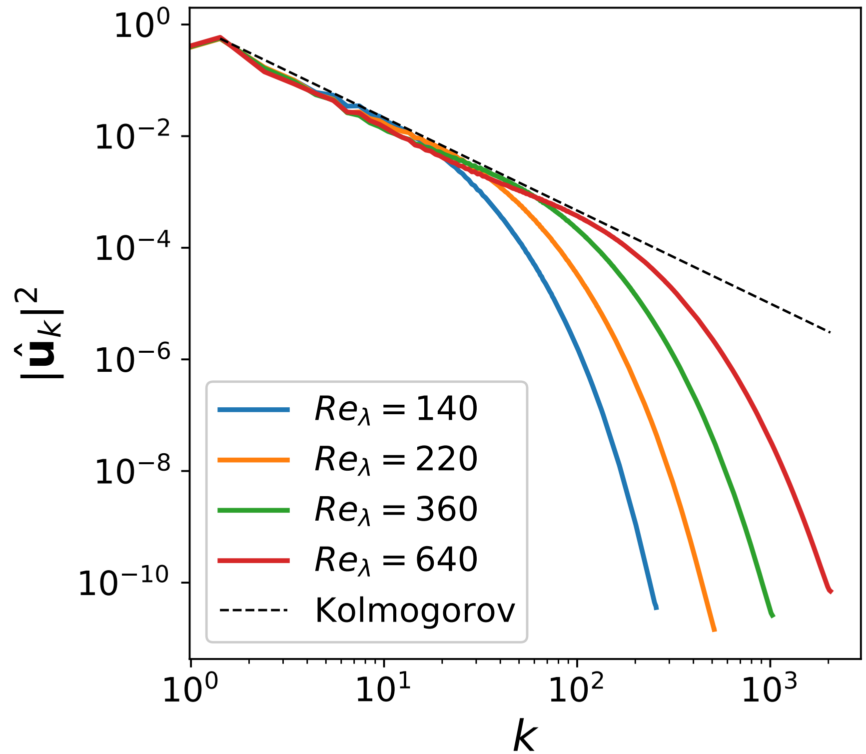
Kinetic energy $|\mathbf{u}|^2/2$



Incompressible, three-dimensional, Navier-Stokes turbulence

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{F}$$



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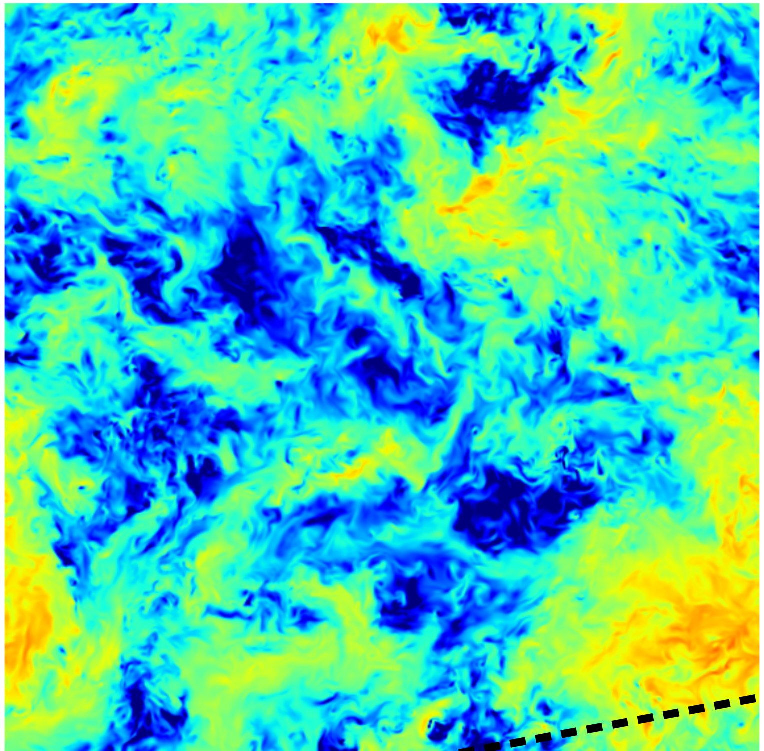

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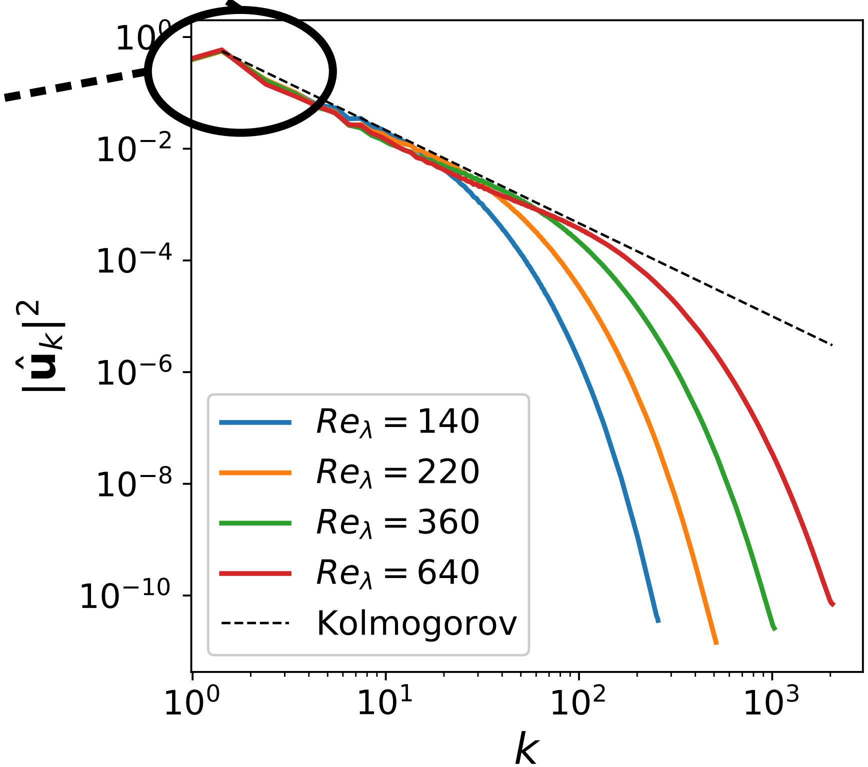
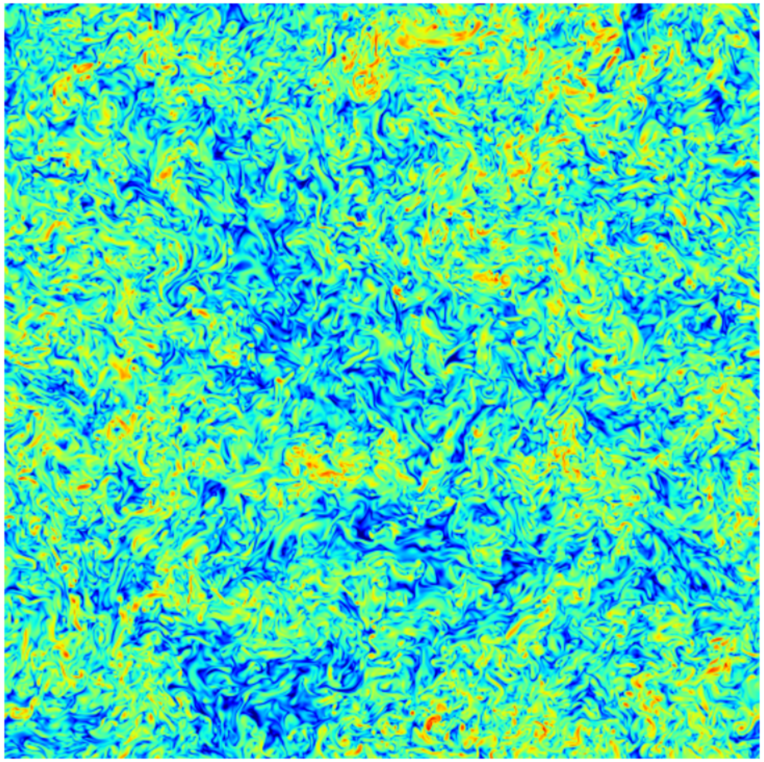


Large scales:

- Forcing influence $\mathbf{F}(t, \boldsymbol{x}; \mathcal{L}, \mathcal{T})$
- Statistically non universal

Velocity gradients describe the small scales

Enstrophy $|\mathbf{W}|^2$



Velocity gradients, strain and rotation rates

$$\mathbf{A} = \nabla \mathbf{u}$$

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^\top\right)$$

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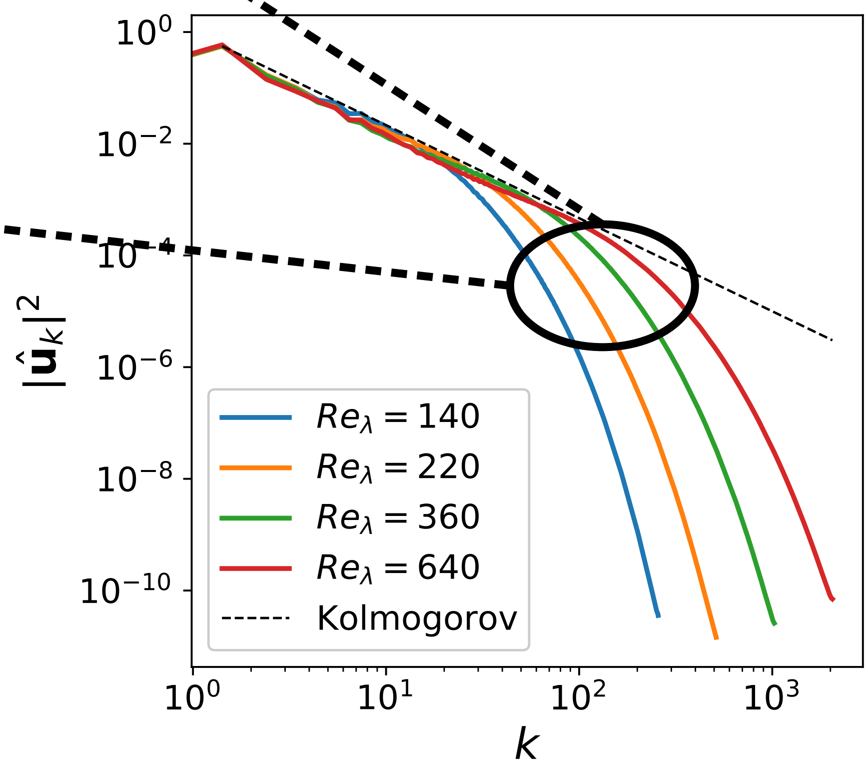

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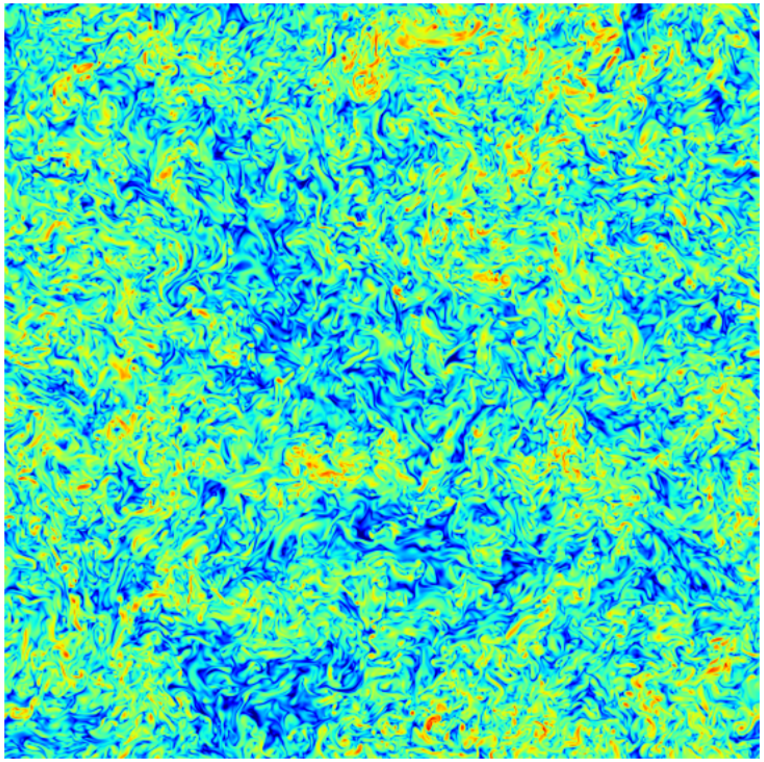

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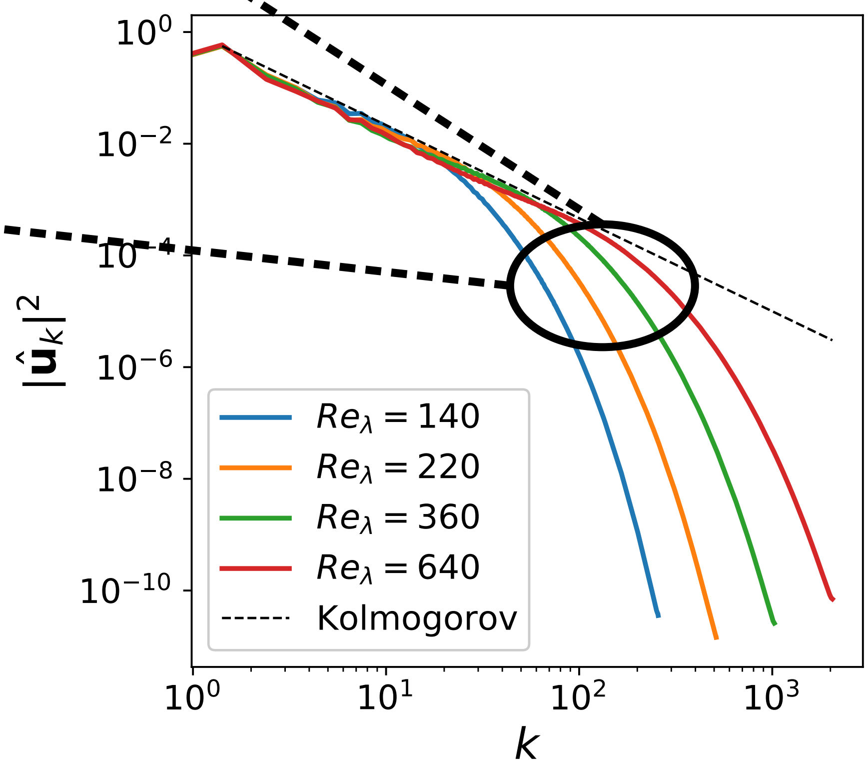
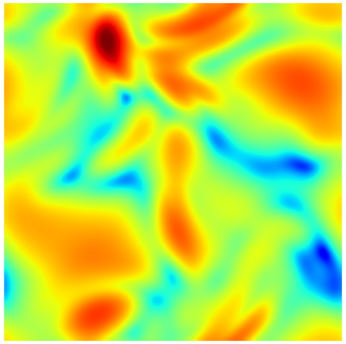


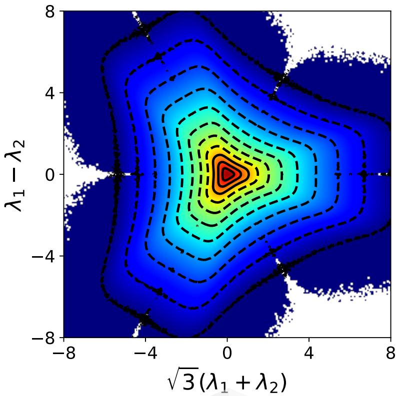
Small scales:

- Forcing fades out
  $\sim 1/Re^p$ **[E. Novikov, FDR, 1993]**
- Statistically universal
- Sustained by direct cascades

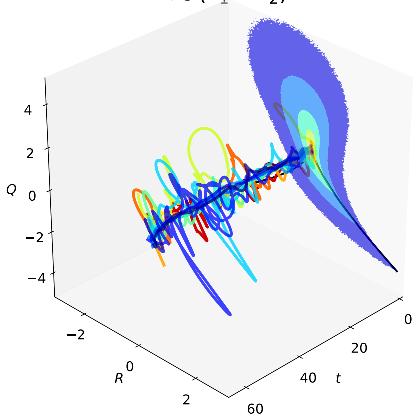


Velocity gradients at low Reynolds numbers

Acknowledgments: Prof. Michael Wilczek



Strain rate at high Reynolds numbers



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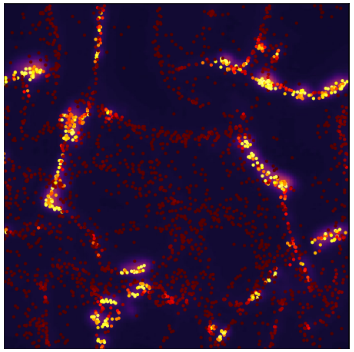

Some applications

Velocity gradients and small-scale turbulence

$$\boldsymbol{A} = \boldsymbol{\nabla} \boldsymbol{u}$$

- Geometry: strain and rotation rates

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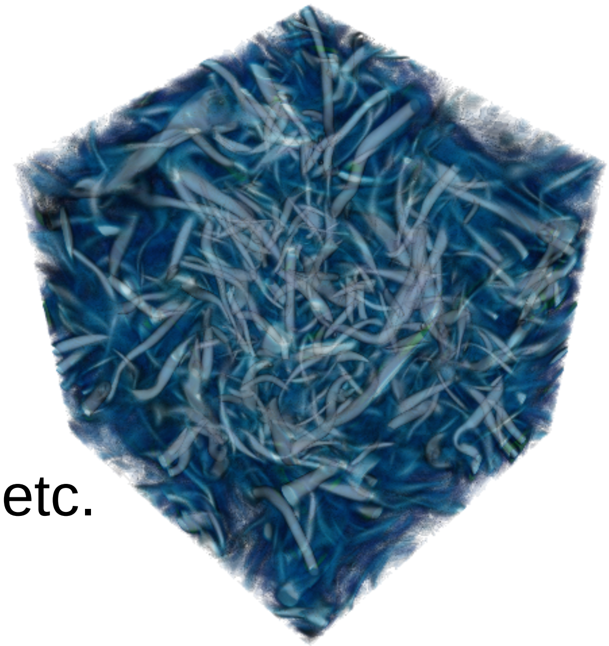

- Invariants: dissipation rate, enstrophy, etc.

$$\varepsilon = 2\mathrm{Tr}\left(\boldsymbol{S}^2\right),\ \omega^2 = -2\mathrm{Tr}\left(\boldsymbol{W}^2\right)$$

Re

Re=*0:* Gaussian **F** ⟶ Gaussian **u**

Re>>*1:* Fully developed turbulence

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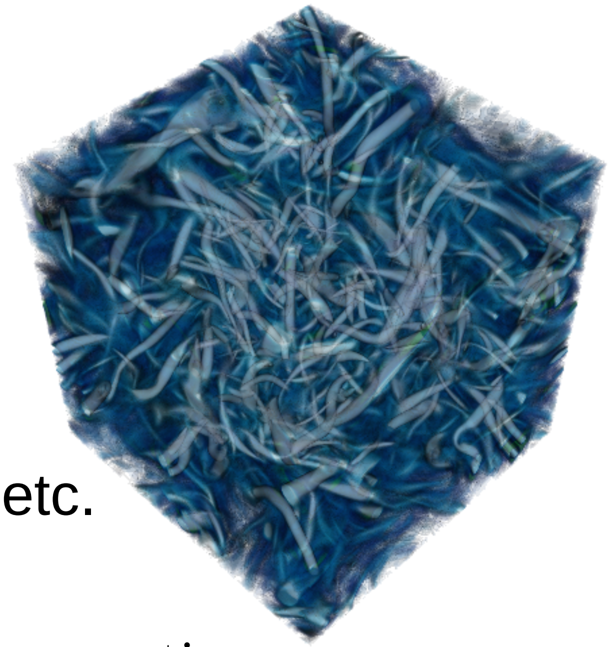

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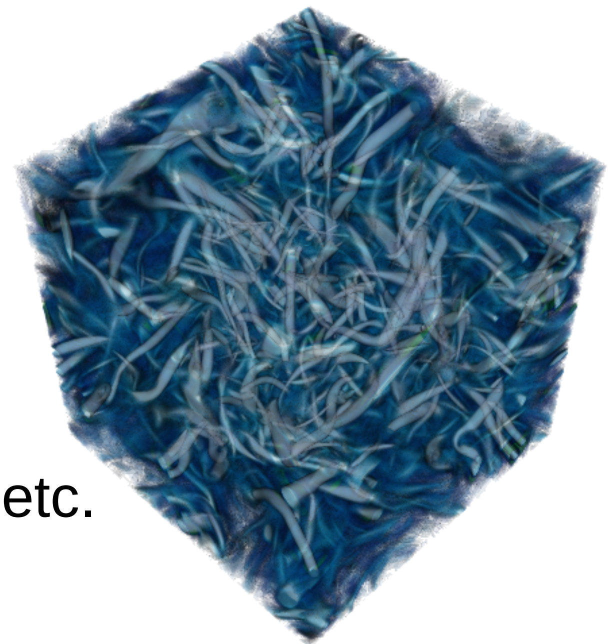

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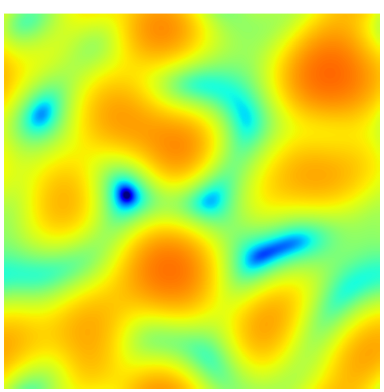
2D slice of the 3D dissipation-rate field at increasing Reynolds



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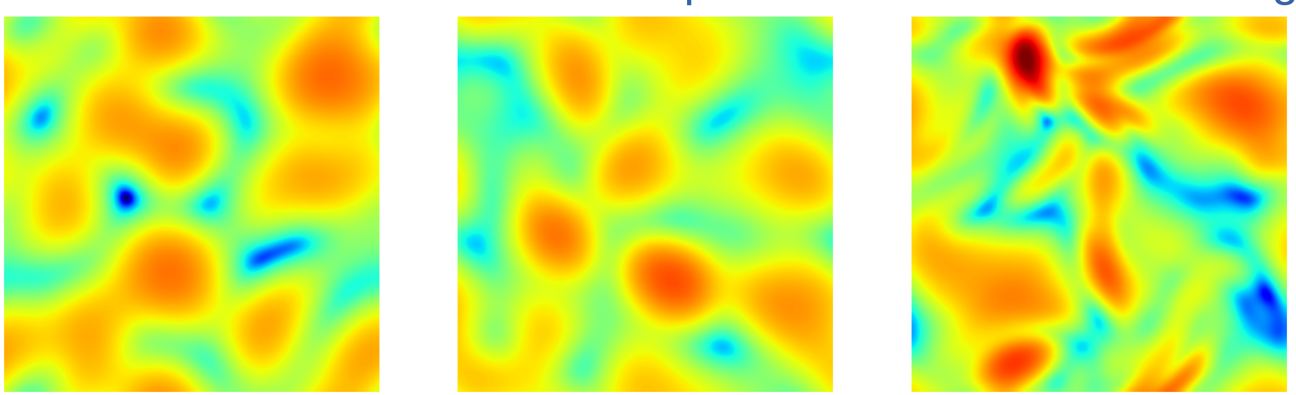

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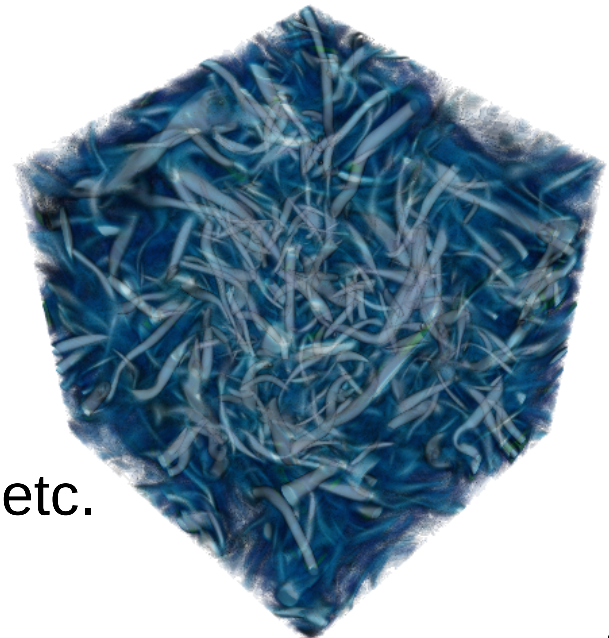

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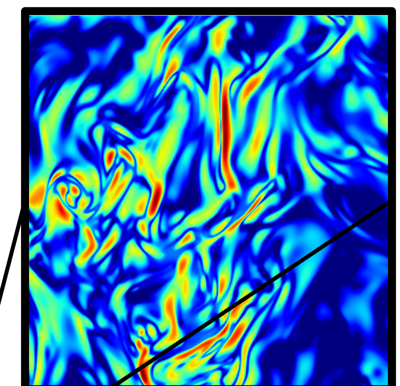

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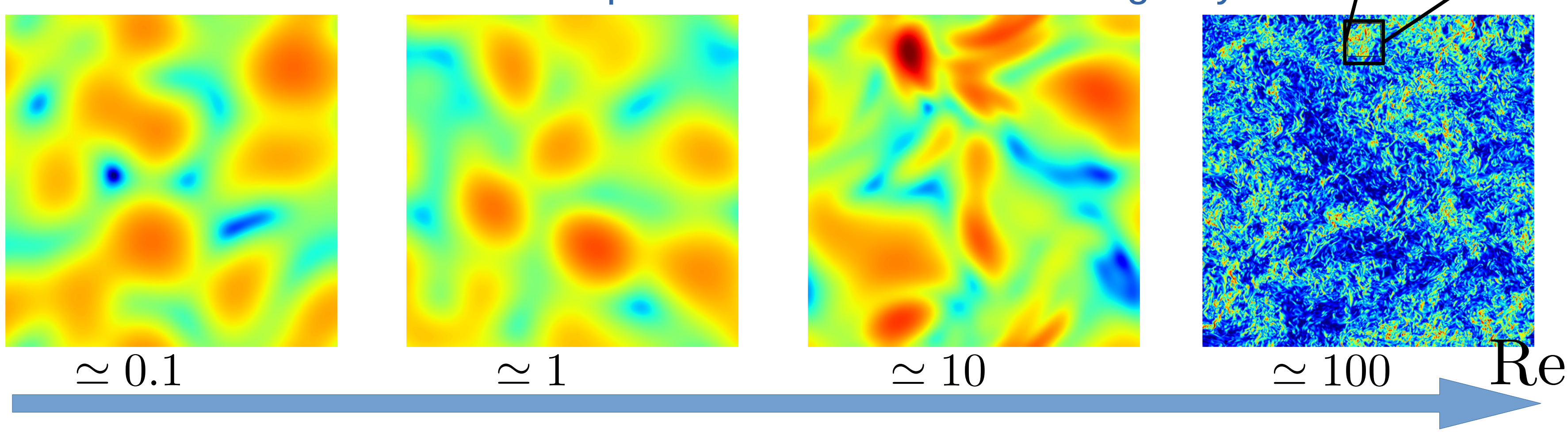

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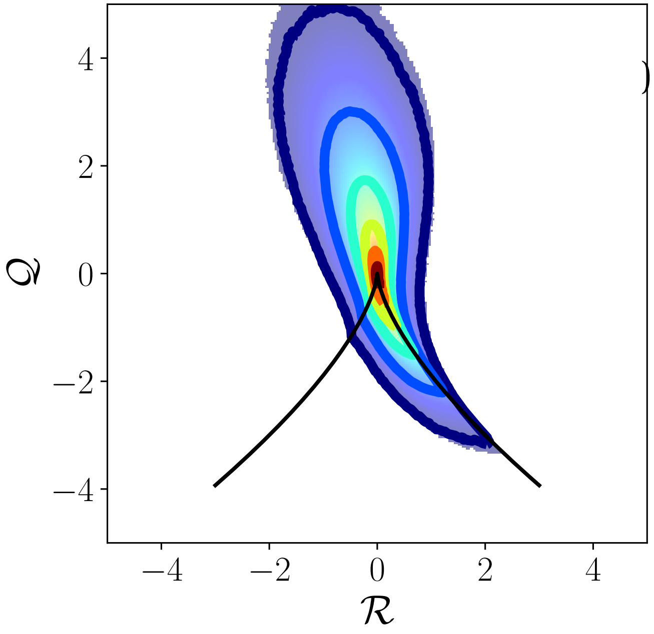
Features of fully developed turbulence

- Skewness, cascades

$$\left\langle \mathrm{Tr}(\boldsymbol{S}^3) \right\rangle < 0$$



$$\mathcal{Q} = -\mathrm{Tr}\left(\boldsymbol{A}^2\right)/2$$

$$\mathcal{R} = -\mathrm{Tr}\left(\boldsymbol{A}^3\right)/3$$

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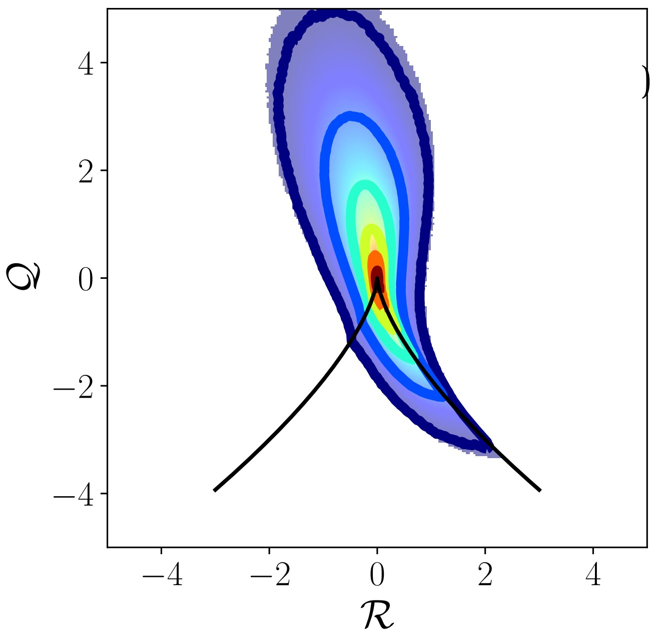

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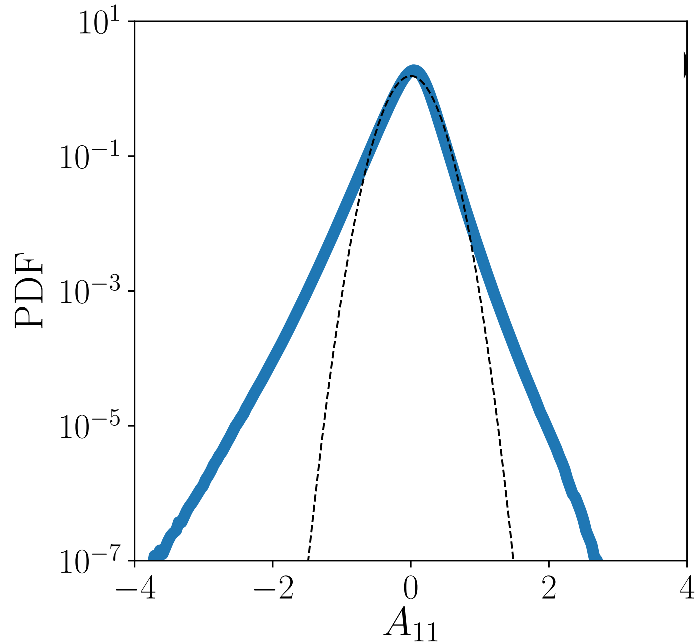
- Intermittency, anomalous scaling

$$\left\langle A_{11}^4\right\rangle \gg 3\left\langle A_{11}^2\right\rangle^2$$



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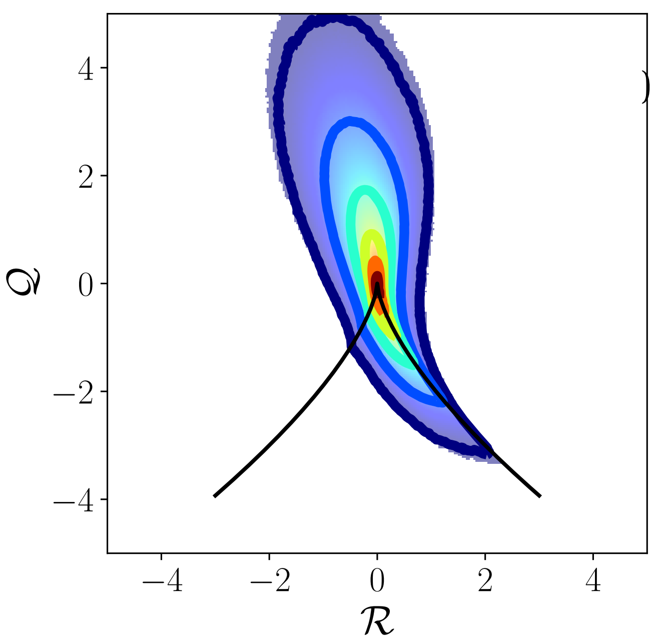

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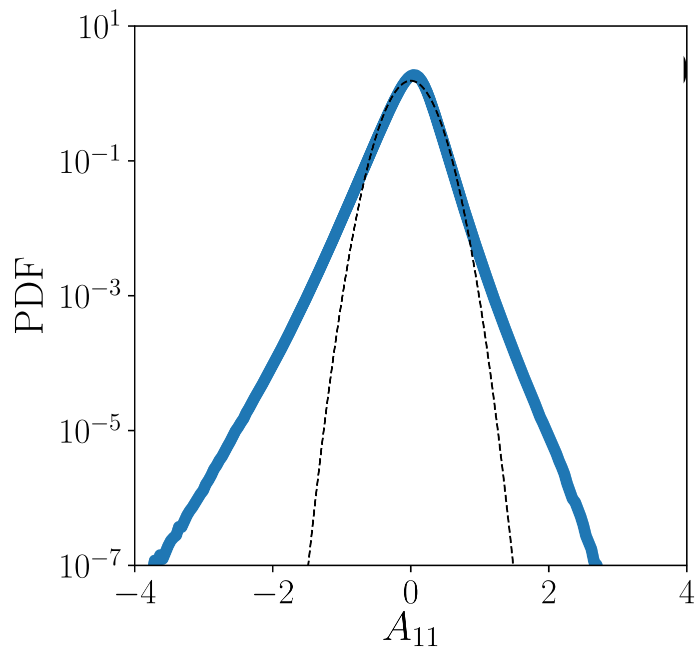
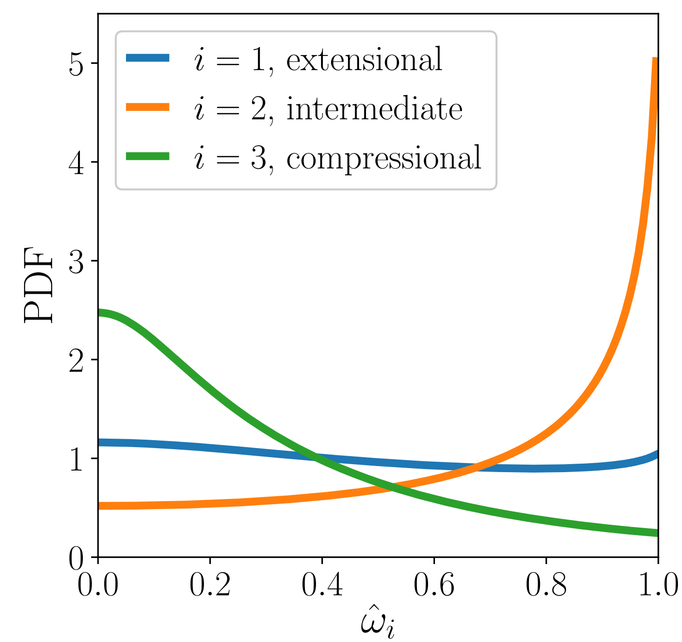


- Alignments strain rate-vorticity

$$\left\langle \mathrm{Tr}\left(\boldsymbol{S}\boldsymbol{W}^2\right) \right\rangle > 0$$



$$\hat{\omega}_i = \frac{\boldsymbol{\omega} \cdot \boldsymbol{v}_i}{\|\boldsymbol{\omega}\|}$$

The onset of fully developed turbulence

- Do low-Reynolds flows exhibit any of the features of high-Reynolds turbulence? [1,2]
- How do the skewness, intermittency, alignments, etc. establish as Reynolds increases?

[1] Yakhot and Donzis, Phys. Rev. Lett., (2017)
[2] Gotoh and Yang, Philos. Trans. Royal Soc. A, (2022)

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- Wyld expansion of the Navier-Stokes equations [3]
- Velocity gradient modelling [4,5]

[1] Yakhot and Donzis, Phys. Rev. Lett., (2017)
[2] Gotoh and Yang, Philos. Trans. Royal Soc. A, (2022)
[3] Wyld, Ann.Phys, (1961)
[4] Meneveau, Annu. Rev. Fluid Mech, (2011)
[5] Leppin and M. Wilczek, Phys. Rev. Lett., (2020)

Randomly-driven Navier-Stokes: from Gaussian field to turbulence

Reynolds number

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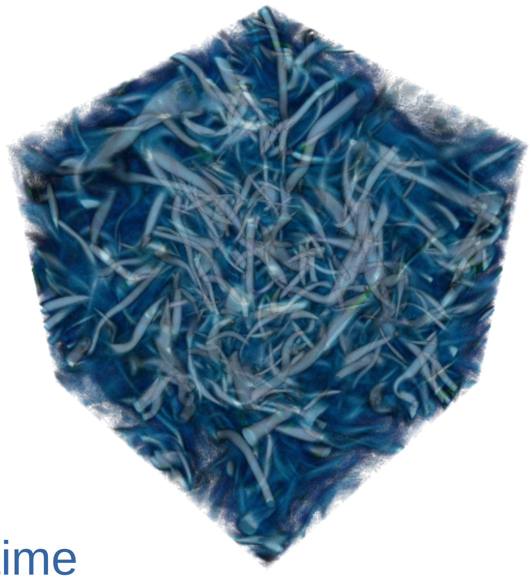
Nonlinear convective term



Velocity field
$\boldsymbol{u}(\boldsymbol{x}, t),\ \boldsymbol{u}, \boldsymbol{x} \in \mathbb{R}^3$

Gaussian forcing:
large scales, white in time

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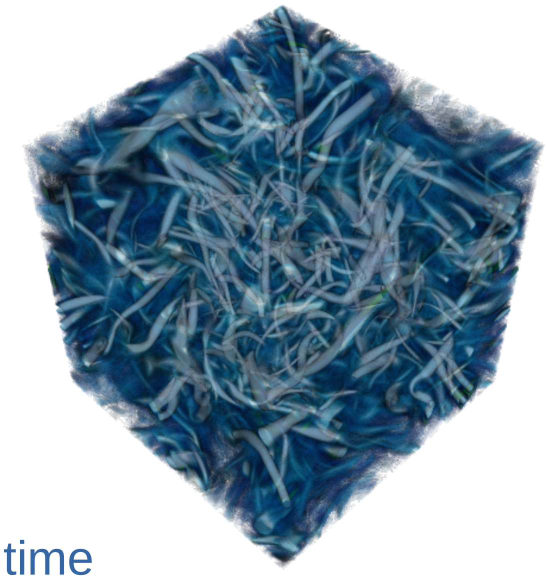

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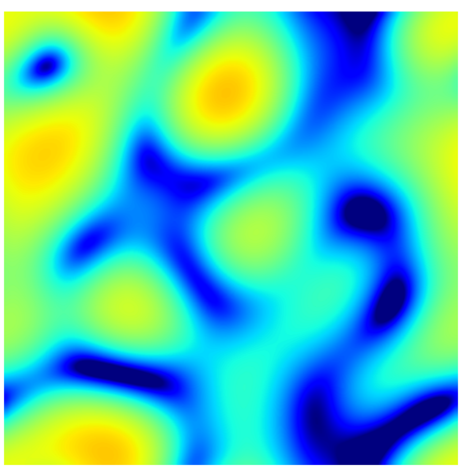

$\simeq 0.1$ $\quad\longrightarrow\quad$ Re

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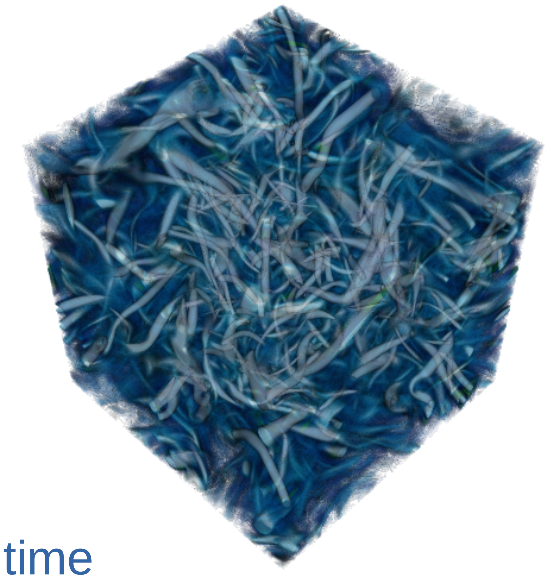

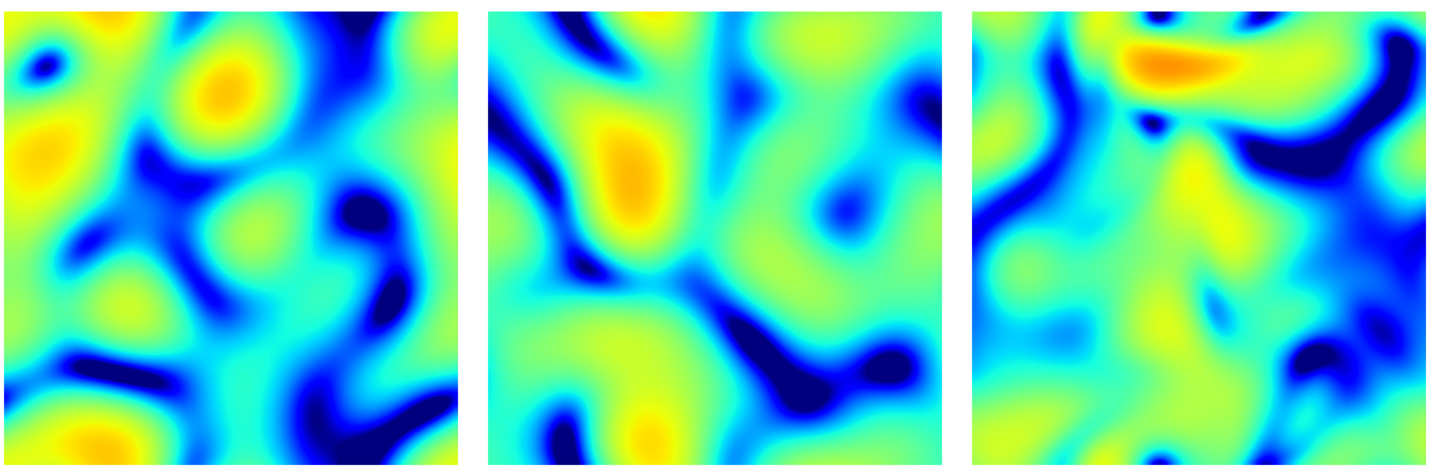

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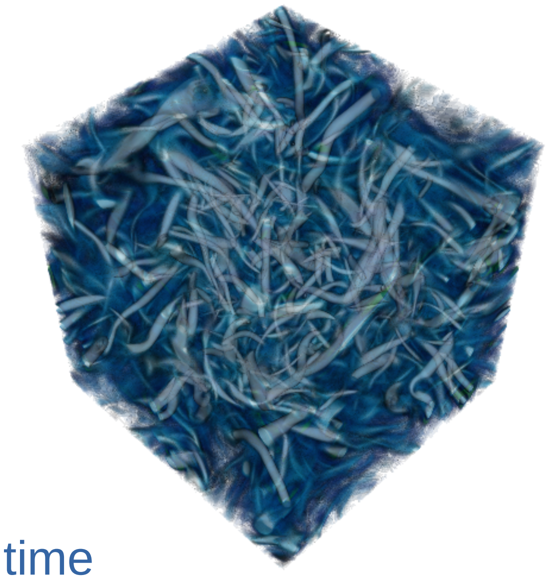

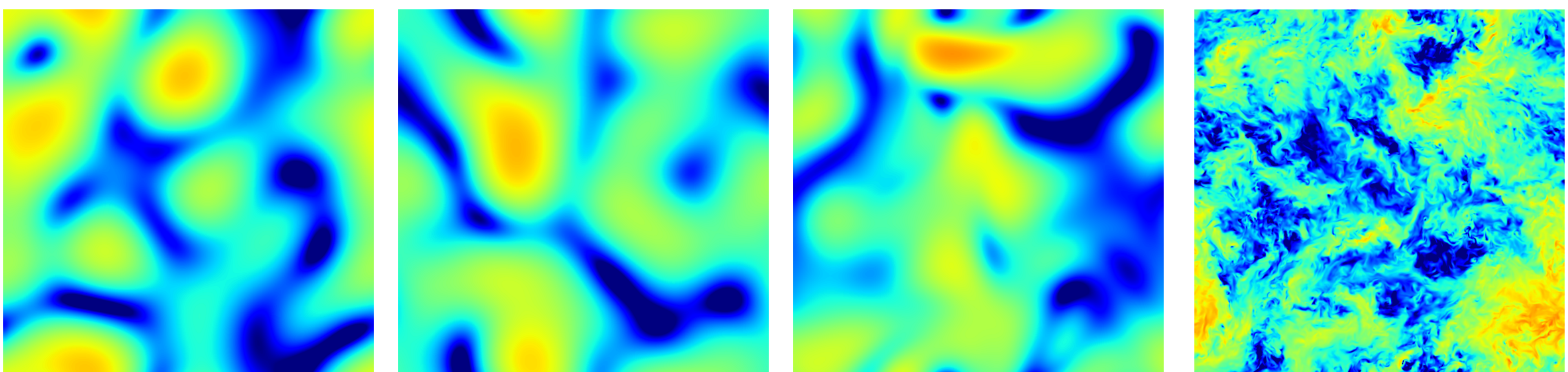

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From the velocity field to the Lagrangian modelling of the velocity gradient

- Langevin for ensemble that shares the same ***A***

$$\mathrm{Tr}(\boldsymbol{A}) = 0$$

unclosed

unclosed

$$\mathrm{d}\boldsymbol{A} = \mathrm{Re}\left[-\widetilde{\boldsymbol{A}^2} - \left\langle \widetilde{\boldsymbol{H}} \middle| \boldsymbol{A} \right\rangle\right]\mathrm{d}t + \langle \nabla^2 \boldsymbol{A} | \boldsymbol{A} \rangle \mathrm{d}t + \sigma \boldsymbol{\nabla} d\boldsymbol{F}$$

Self interaction, pressure Hessian, viscous Laplacian, Gaussian forcing

- Fewer degrees of freedom: unclosed terms, modelling!

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Self interaction, pressure Hessian, viscous Laplacian, Gaussian forcing

- Fewer degrees of freedom: unclosed terms, modelling!

- Re=*0*: Gaussian, known Hessian and viscous terms

$$\left\langle \widetilde{\boldsymbol{H}} \middle| \boldsymbol{A} \right\rangle = -\frac{2}{7}\widetilde{\boldsymbol{S}^2} - \frac{2}{5}\widetilde{\boldsymbol{W}^2} + \mathcal{O}(\mathrm{Re})$$

$$\left\langle \nabla^2 \boldsymbol{A} \middle| \boldsymbol{A} \right\rangle = -\gamma_0 \boldsymbol{A} + \mathcal{O}(\mathrm{Re})$$

- Pressure Hessian: exact at first order
- Viscous corrections: to be modeled

Modelling through tensor function representation

Modeling

$$\left\langle -\mathrm{Re}\widetilde{\boldsymbol{H}} + \nabla^2 \boldsymbol{A} \middle| \boldsymbol{A} \right\rangle = \sum_{n=1}^{8} \gamma_n \boldsymbol{B}_n(\boldsymbol{A})$$

- Basis tensors: second order in ***A***
- Constant coefficients γ_n

Wyld zeroth-order expansion

$$\mathrm{d}\boldsymbol{A} = -\gamma_0 \boldsymbol{A}\mathrm{d}t + \sigma d\boldsymbol{\nabla F} + \\ + \mathrm{Re}\left[\mathrm{Re}\delta_1 \boldsymbol{S} + \mathrm{Re}\delta_2 \boldsymbol{W} + \left(\delta_3 - \frac{5}{7}\right)\widetilde{\boldsymbol{S}^2} + \right. \\ \left. + (\delta_5 - 1)(\boldsymbol{SW} + \boldsymbol{WS}) + \left(\delta_6 - \frac{3}{5}\right)\widetilde{\boldsymbol{W}^2}\right]\mathrm{d}t + \text{Gauge}$$

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Constraints

Unity time scale

$$\left\langle \mathrm{Tr}\left(\boldsymbol{S}^2\right)\right\rangle = \frac{1}{2}$$

Homogeneity

$$\left\langle \mathrm{Tr}\left(\boldsymbol{A}^2\right)\right\rangle = 0 \quad \left\langle \mathrm{Tr}\left(\boldsymbol{A}^3\right)\right\rangle = 0$$

Wyld, weak coupling

$$\left\langle \mathrm{Tr}\left(\boldsymbol{S}^3\right)\right\rangle = S_3 \mathrm{Re}$$

$$\left\langle \mathrm{Tr}\left(\boldsymbol{S}^2\boldsymbol{W}^2\right)\right\rangle = -\frac{1}{12} + X_5 \mathrm{Re}^2$$

Wyld zeroth-order expansion

$$\begin{aligned} \mathrm{d}\boldsymbol{A} = &-\boxed{\gamma_0}\boldsymbol{A}\mathrm{d}t + \boxed{\sigma} d\boldsymbol{\nabla F} + \\ &+ \mathrm{Re}\left[\mathrm{Re}\delta_1\boldsymbol{S} + \mathrm{Re}\delta_2\boldsymbol{W} + \left(\delta_3 - \frac{5}{7}\right)\widetilde{\boldsymbol{S}^2} + \right. \\ &\left. + \left(\delta_5 - 1\right)\left(\boldsymbol{S}\boldsymbol{W} + \boldsymbol{W}\boldsymbol{S}\right) + \left(\delta_6 - \frac{3}{5}\right)\widetilde{\boldsymbol{W}^2}\right]\mathrm{d}t + \mathrm{Gauge} \end{aligned}$$

Solvable Fokker-Planck Equation

Velocity gradient PDF parametrized through the invariants

Polynomial coefficients

$$\alpha(\mathcal{I})f(\mathcal{I}) + v_k(\mathcal{I})\frac{\partial f}{\partial \mathcal{I}_k}(\mathcal{I}) - D_{jk}(\mathcal{I})\frac{\partial^2 f}{\partial \mathcal{I}_j \partial \mathcal{I}_k}(\mathcal{I}) = 0$$

$$\mathcal{I}_1 = \mathrm{Tr}\left(\boldsymbol{S}^2\right) \quad \mathcal{I}_2 = \mathrm{Tr}\left(\boldsymbol{W}^2\right)$$

$$\mathcal{I}_3 = \mathrm{Tr}\left(\boldsymbol{S}^3\right) \quad \mathcal{I}_4 = \mathrm{Tr}\left(\boldsymbol{S}\boldsymbol{W}^2\right)$$

$$\mathcal{I}_5 = \mathrm{Tr}\left(\boldsymbol{S}^2\boldsymbol{W}^2\right)$$

Independent invariants

Solvable Fokker-Planck Equation

Velocity gradient PDF parametrized through the invariants

Polynomial coefficients

$$\alpha(\mathcal{I}) f(\mathcal{I}) + v_k(\mathcal{I}) \frac{\partial f}{\partial \mathcal{I}_k}(\mathcal{I}) - D_{jk}(\mathcal{I}) \frac{\partial^2 f}{\partial \mathcal{I}_j \partial \mathcal{I}_k}(\mathcal{I}) = 0$$

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Independent invariants

Asymptotic solution:
Polynomial *x* Gaussian

$$\begin{aligned}
f = {} & \frac{225\sqrt{5}}{\pi^4} e^{-5\mathcal{I}_1 + 3\mathcal{I}_2} + \\
& + \mathrm{Re}_\gamma \frac{3600\sqrt{5} S_3 \left(25\mathcal{I}_3 - 21\mathcal{I}_4\right)}{7\pi^4} e^{-5\mathcal{I}_1 + 3\mathcal{I}_2} + \\
& + \mathrm{Re}_\gamma^2 \frac{720\sqrt{5}}{49\pi^4} \big(- 16320 S_3^2 \mathcal{I}_1 \mathcal{I}_2 - 6860 S_3^2 \mathcal{I}_1 - 1344 S_3^2 \mathcal{I}_2^2 - 140 S_3^2 \mathcal{I}_2 + 50000 S_3^2 \mathcal{I}_3^2 + \\
& - 84000 S_3^2 \mathcal{I}_3 \mathcal{I}_4 + 35280 S_3^2 \mathcal{I}_4^2 + 42240 S_3^2 \mathcal{I}_5 + 2240 S_3^2 - 22950 X_5 \mathcal{I}_1 \mathcal{I}_2 - 1575 X_5 \mathcal{I}_1 + \\
& - 1890 X_5 \mathcal{I}_2^2 - 1575 X_5 \mathcal{I}_2 + 59400 X_5 \mathcal{I}_5 \big) e^{-5\mathcal{I}_1 + 3\mathcal{I}_2}.
\end{aligned}$$

Solvable Fokker-Planck Equation

Velocity gradient PDF parametrized through the invariants

Polynomial coefficients

$$\alpha(\mathcal{I})f(\mathcal{I}) + v_k(\mathcal{I})\frac{\partial f}{\partial \mathcal{I}_k}(\mathcal{I}) - D_{jk}(\mathcal{I})\frac{\partial^2 f}{\partial \mathcal{I}_j \partial \mathcal{I}_k}(\mathcal{I}) = 0$$

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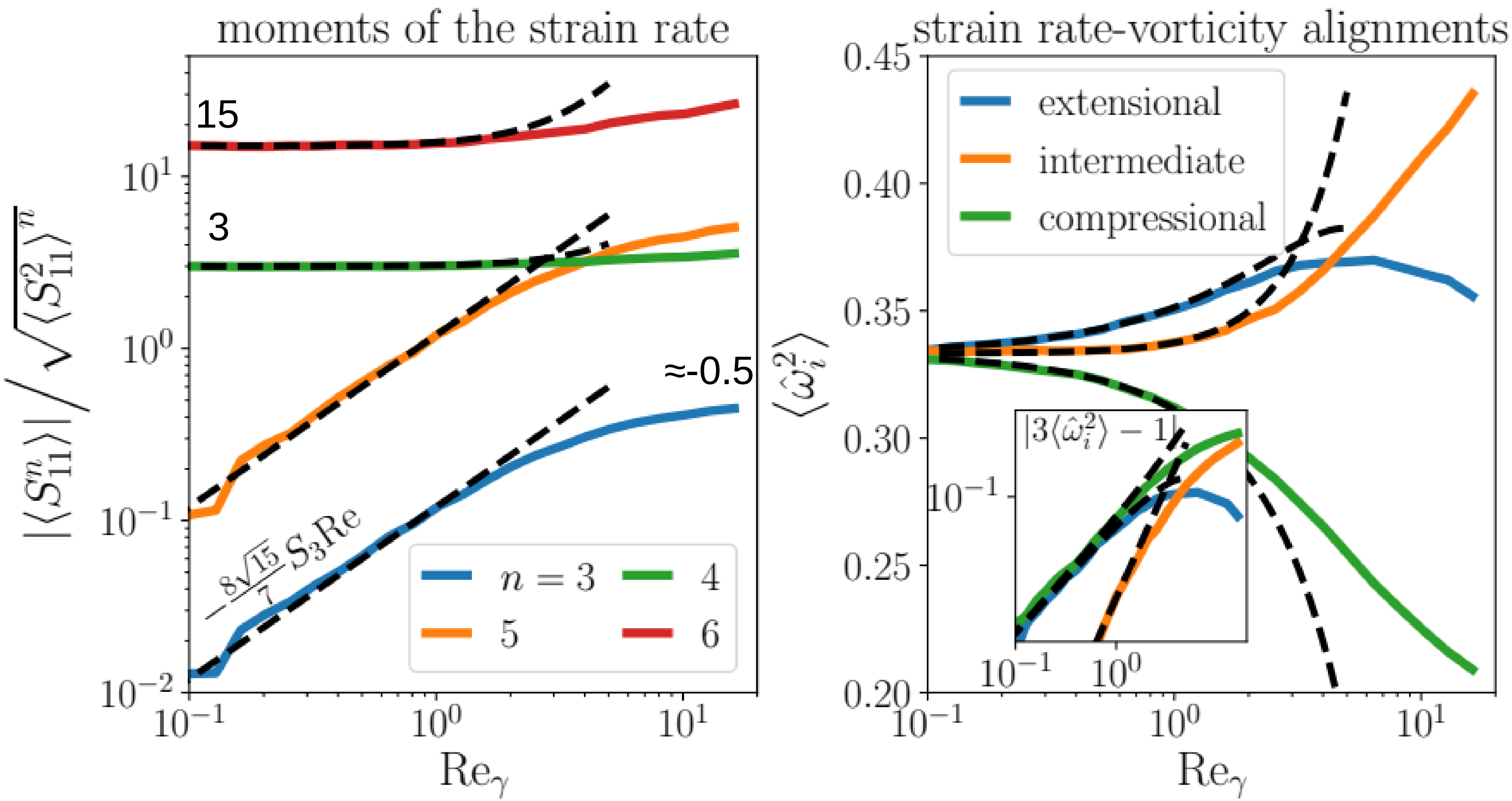
$$- 84000S_3^2\mathcal{I}_3\mathcal{I}_4 + 35280S_3^2\mathcal{I}_4^2 + 42240S_3^2\mathcal{I}_5 + 2240S_3^2 - 22950X_5\mathcal{I}_1\mathcal{I}_2 - 1575X_5\mathcal{I}_1 +$$

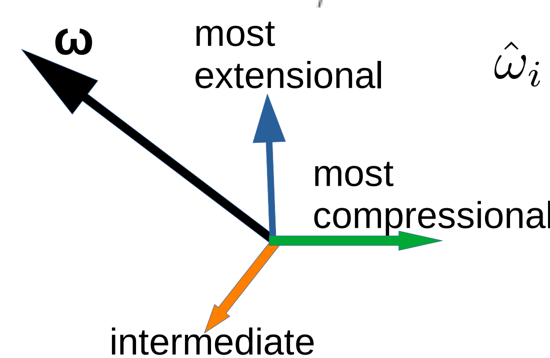
$$- 1890X_5\mathcal{I}_2^2 - 1575X_5\mathcal{I}_2 + 59400X_5\mathcal{I}_5\Big)e^{-5\mathcal{I}_1+3\mathcal{I}_2}.$$

Perturbation parameter

$$\mathrm{Re}_\gamma = \frac{\gamma_0^2}{\nu\tau_\eta} \simeq 1.5\mathrm{Re}_\lambda$$

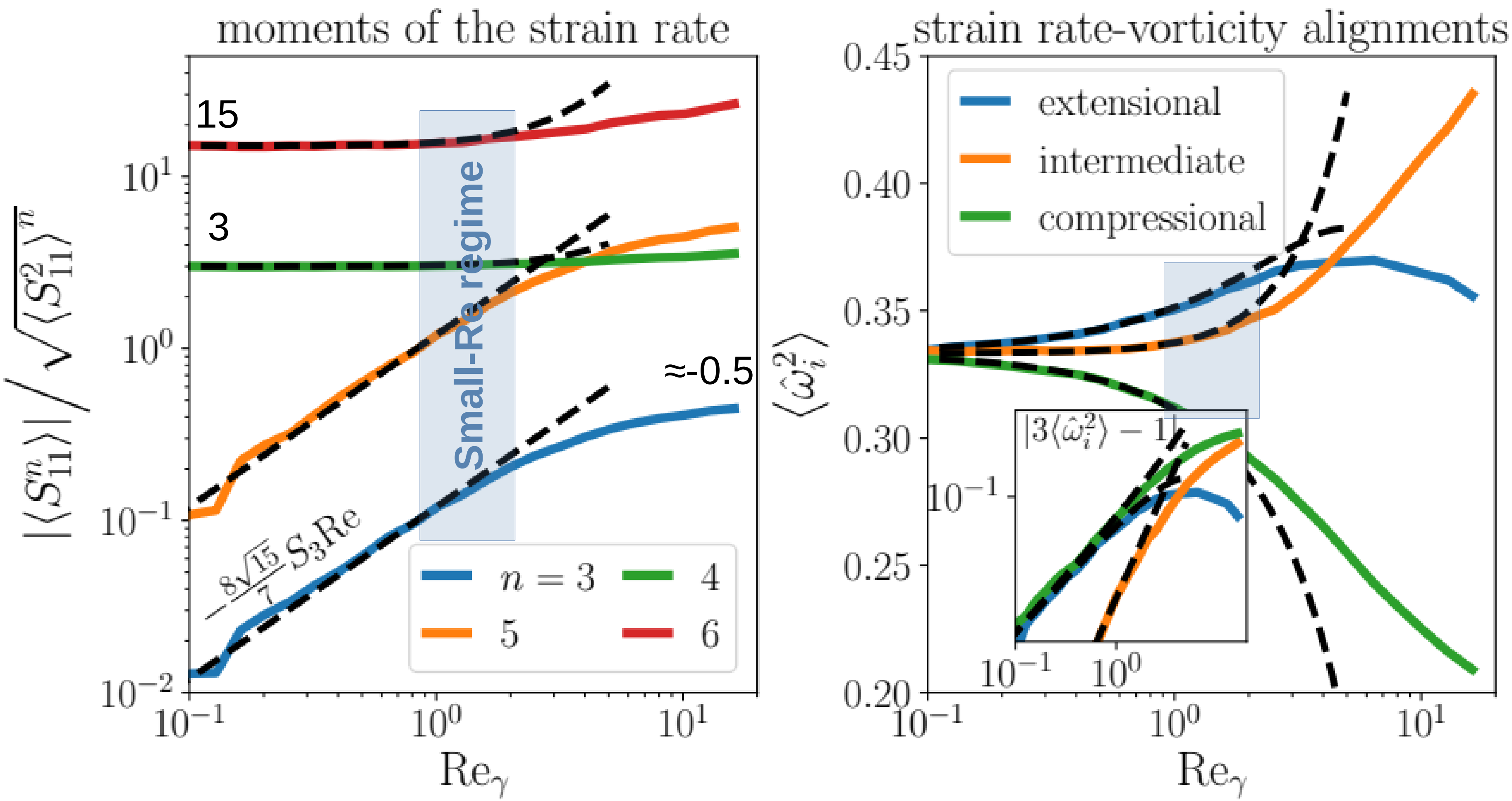
Onset of non-Gaussianity in the velocity gradient statistics



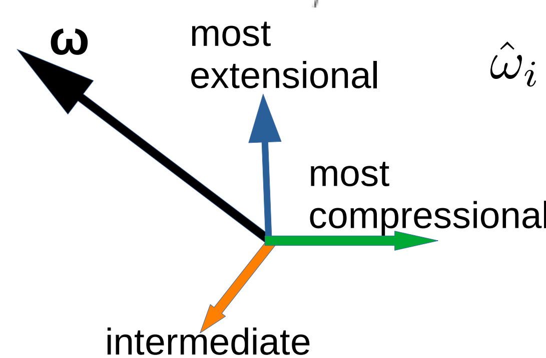


$$\hat{\omega}_i = \frac{\boldsymbol{\omega}\cdot\boldsymbol{v}_i}{\|\boldsymbol{\omega}\|}$$

Onset of non-Gaussianity in the velocity gradient statistics



- At $\mathrm{Re}_\gamma \approx 1$: skewness, negligible intermittency, vorticity aligns with extensional strain-rate eigenvector



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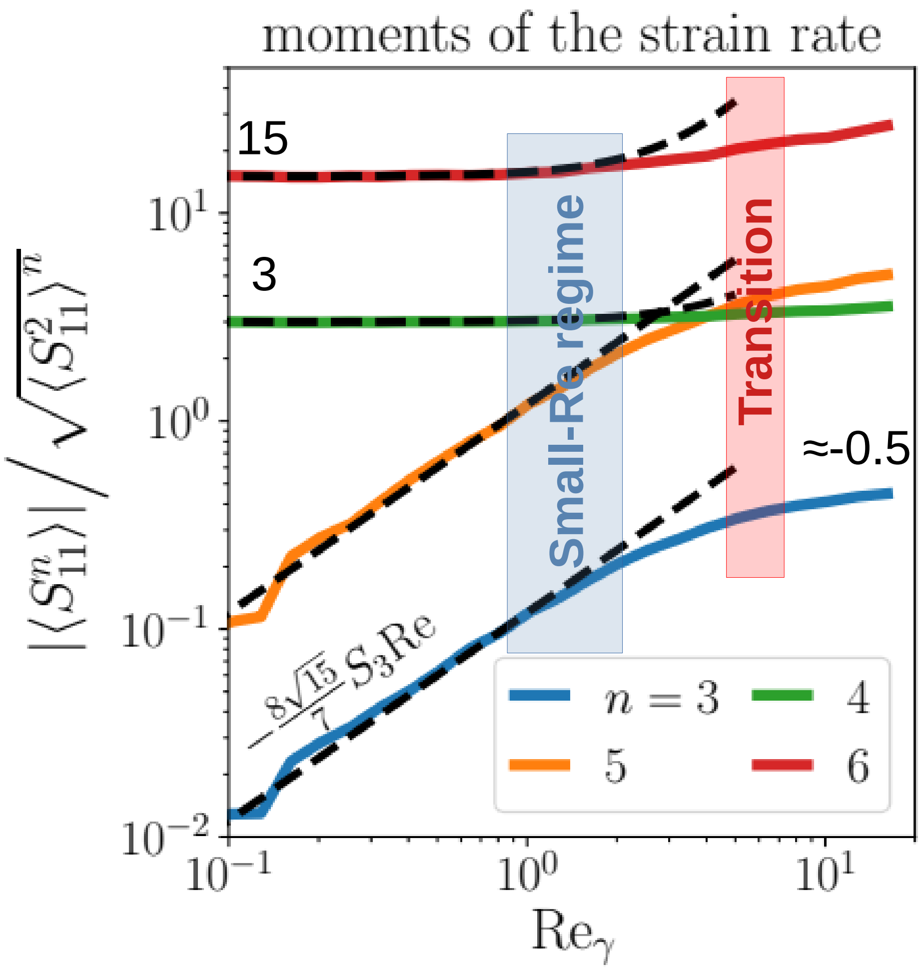
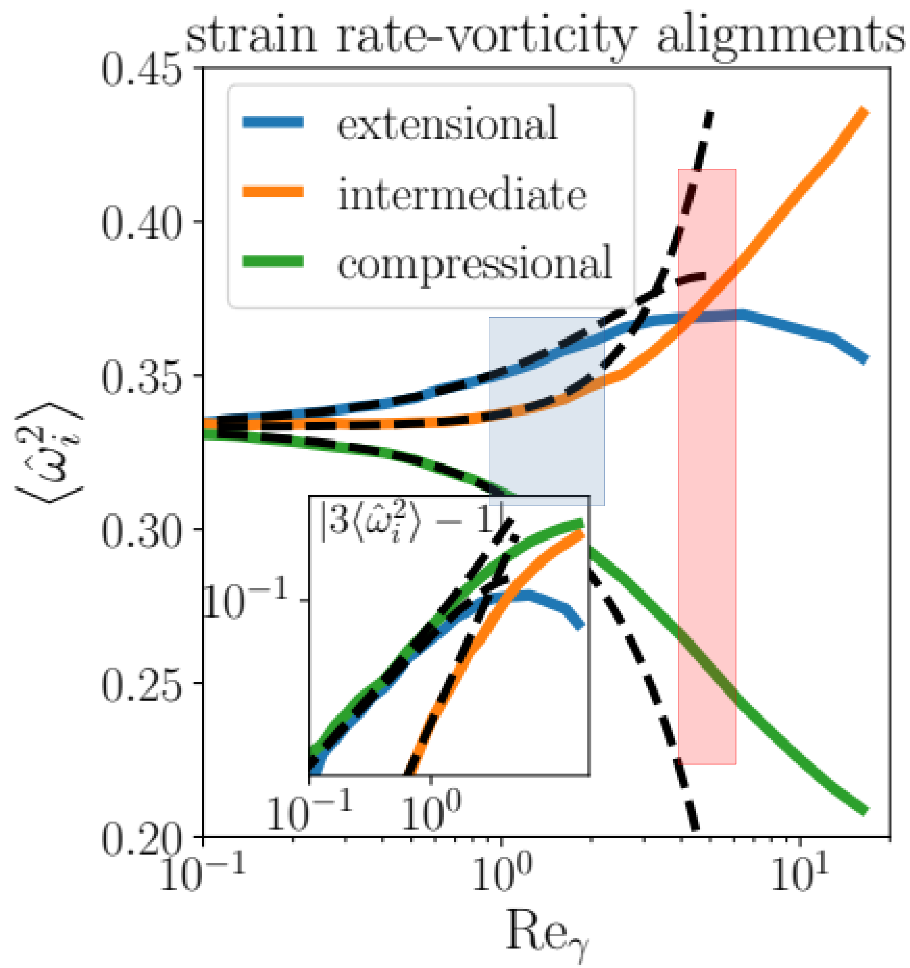
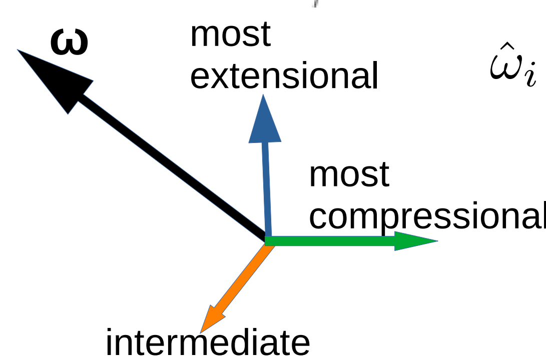
Onset of non-Gaussianity in the velocity gradient statistics







$$\hat{\omega}_i = \frac{\boldsymbol{\omega} \cdot \boldsymbol{v}_i}{\|\boldsymbol{\omega}\|}$$

- At $\mathrm{Re}_\gamma \approx 1$: skewness, negligible intermittency, vorticity aligns with extensional strain-rate eigenvector
- At $\mathrm{Re}_\gamma \approx 5$: smooth transition to turbulence

Skewness in the strain-rate PDF

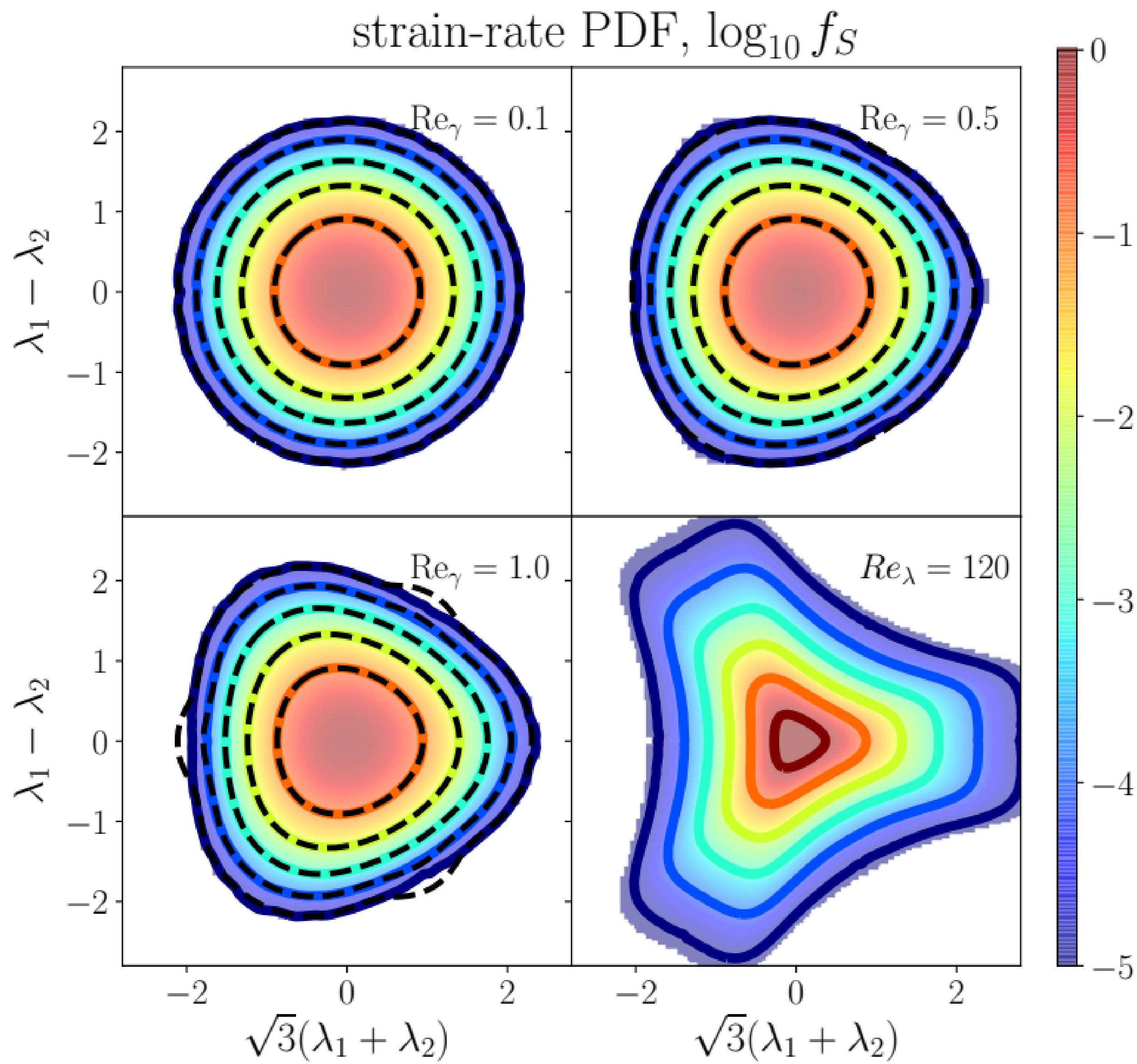
$$\boldsymbol{S} = \sum_{i=1}^{3} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^\top$$

PDF weighted by
Wigner repulsion term J_s

$$f_S(\boldsymbol{S})\mathrm{d}\boldsymbol{S} = f(\lambda) J_S(\lambda) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2$$

$$J_S \propto \prod_{i \neq j} |\lambda_i - \lambda_j|$$

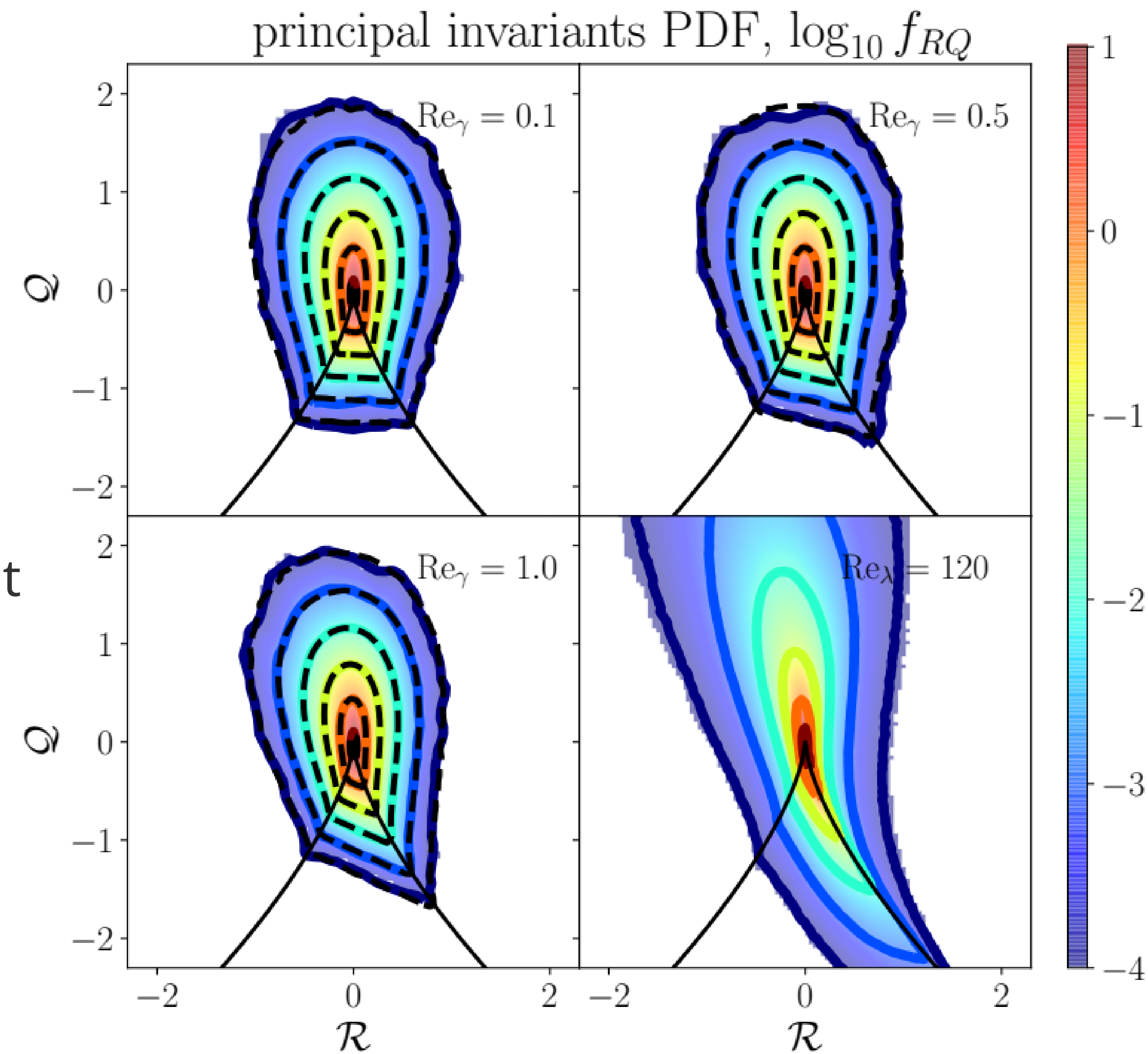
- Two strain-rate eigenvalues are similar..
- ..the other large and negative
- Very simple contours!



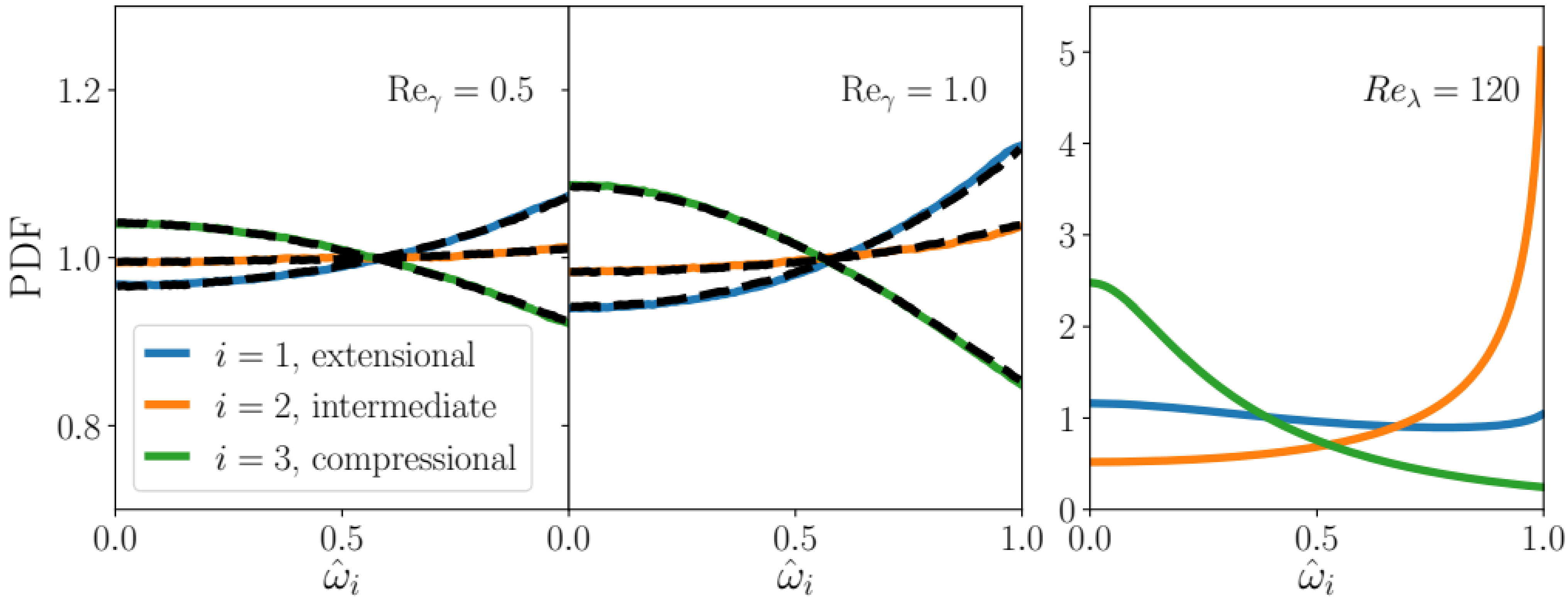
Teardrop PDF of the principal invariants

$$\mathcal{Q} = -\mathrm{Tr}\left(\boldsymbol{A}^2\right)/2$$
$$\mathcal{R} = -\mathrm{Tr}\left(\boldsymbol{A}^3\right)/3$$

- PDF skewed along right Vieillefosse tail
- Intermittency establishes at larger Re



The non-monotonic alignments of the vorticity with the strain rate



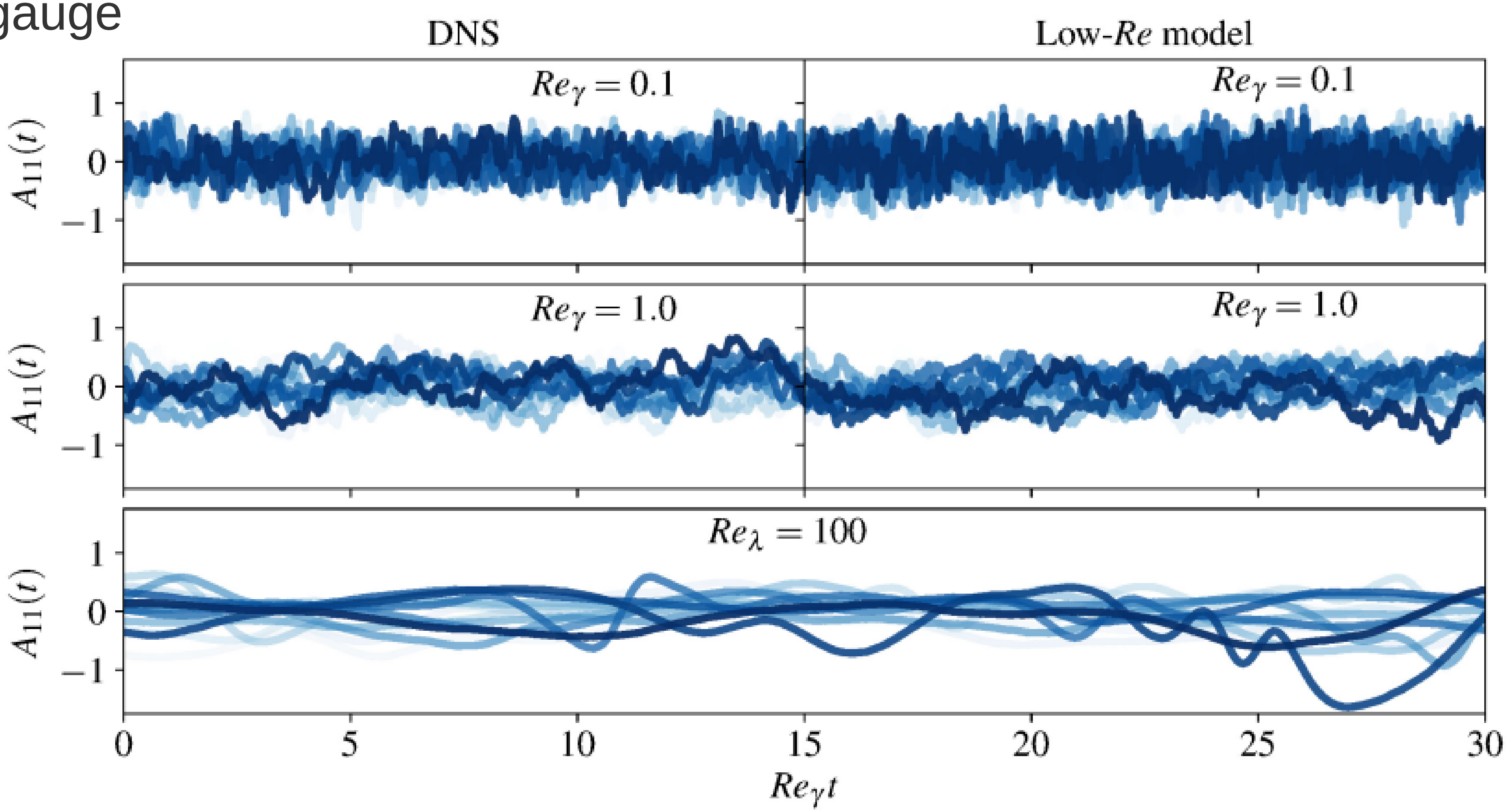
$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \boldsymbol{S} \cdot \boldsymbol{\omega} + \ldots$$

$$\hat{\omega}_i = \frac{\boldsymbol{\omega} \cdot \boldsymbol{v}_i}{\|\boldsymbol{\omega}\|}$$

- Vorticity aligns with extensional direction at small Reynolds
- Alignment with intermediate eigenvector establishes later on

Velocity gradient realizations: DNS and model

- Time correlations through gauge terms



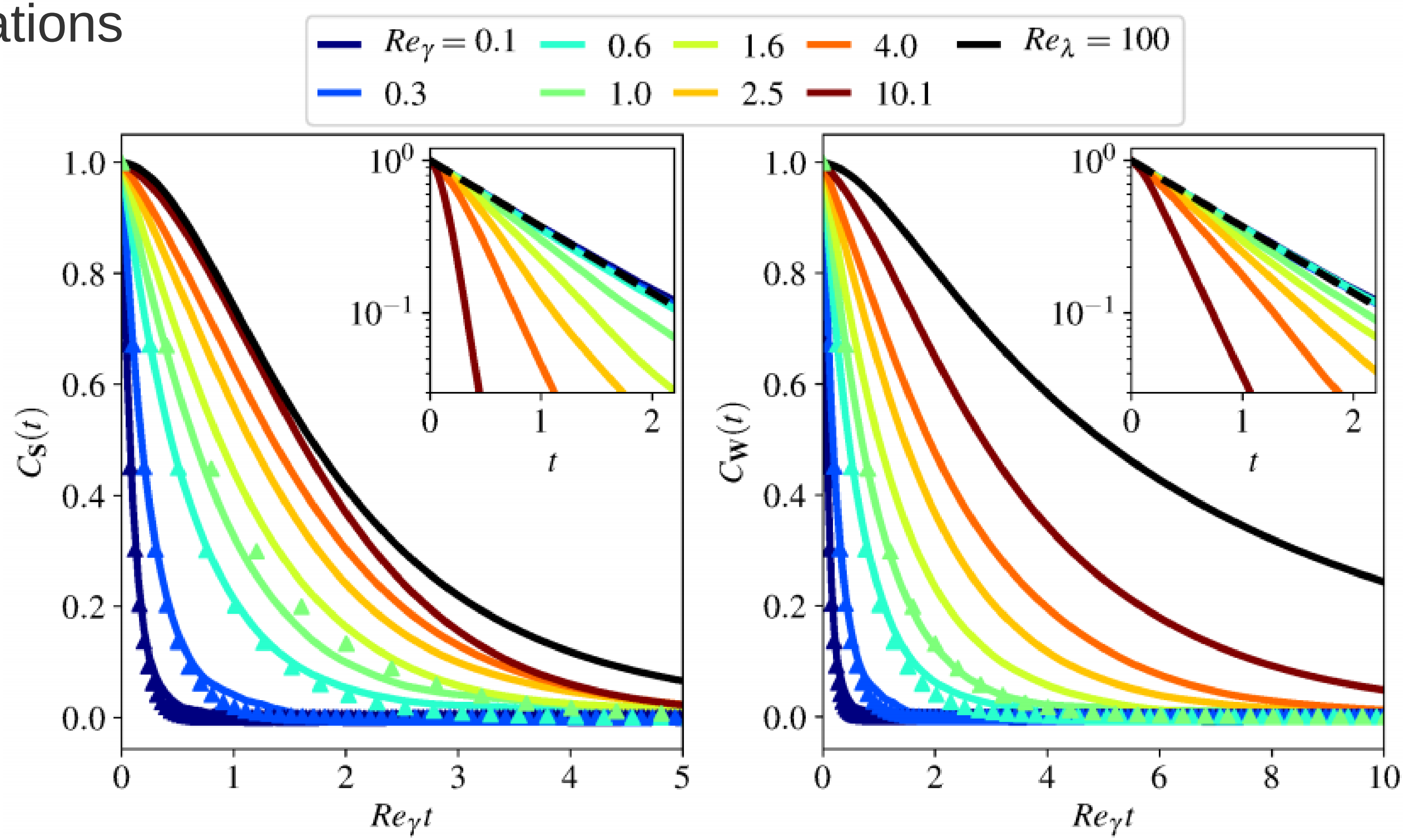
$$\mathrm{Tr}(\boldsymbol{A}) = 0$$

$$\mathrm{d}\boldsymbol{A} = \mathrm{Re}\left[-\widetilde{\boldsymbol{A}}^2 - \left\langle \widetilde{\boldsymbol{H}} \middle| \boldsymbol{A} \right\rangle\right] \mathrm{d}t + \left\langle \nabla^2 \boldsymbol{A} \middle| \boldsymbol{A} \right\rangle \mathrm{d}t + \sigma \boldsymbol{\nabla} d\boldsymbol{F}$$

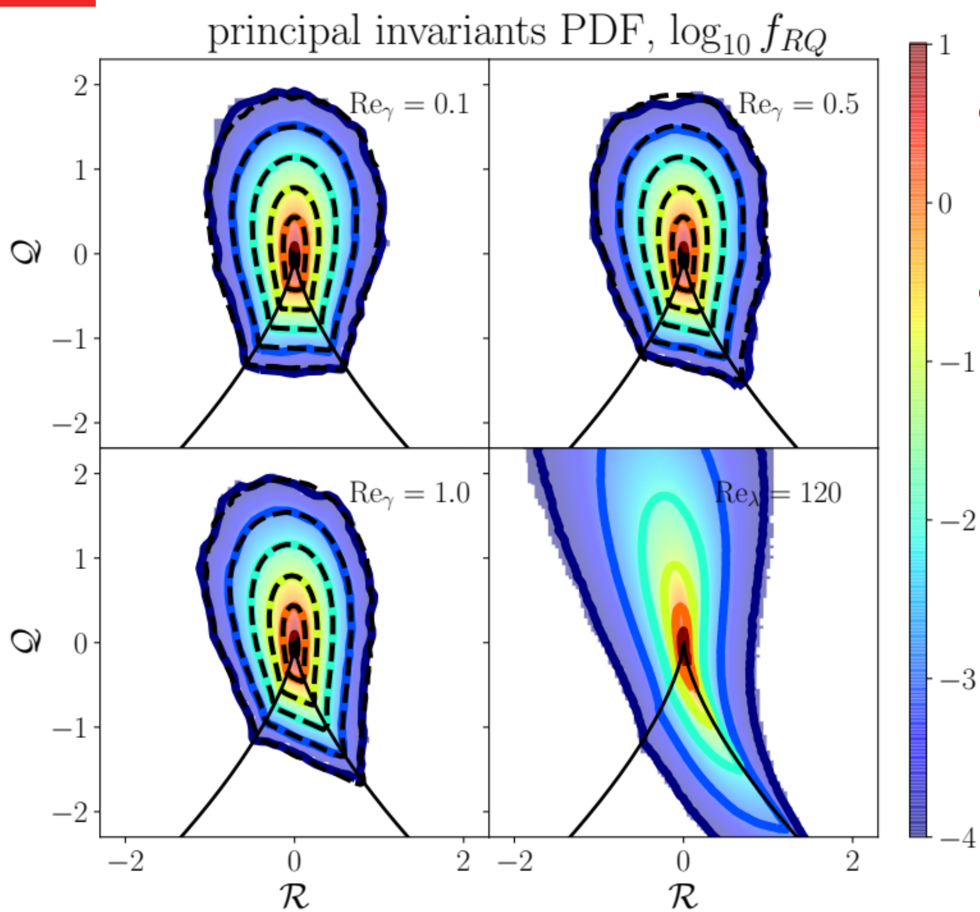
Velocity gradient realizations: time correlations

- Strain-rate correlations at small Re
- Turbulence hinders time correlations

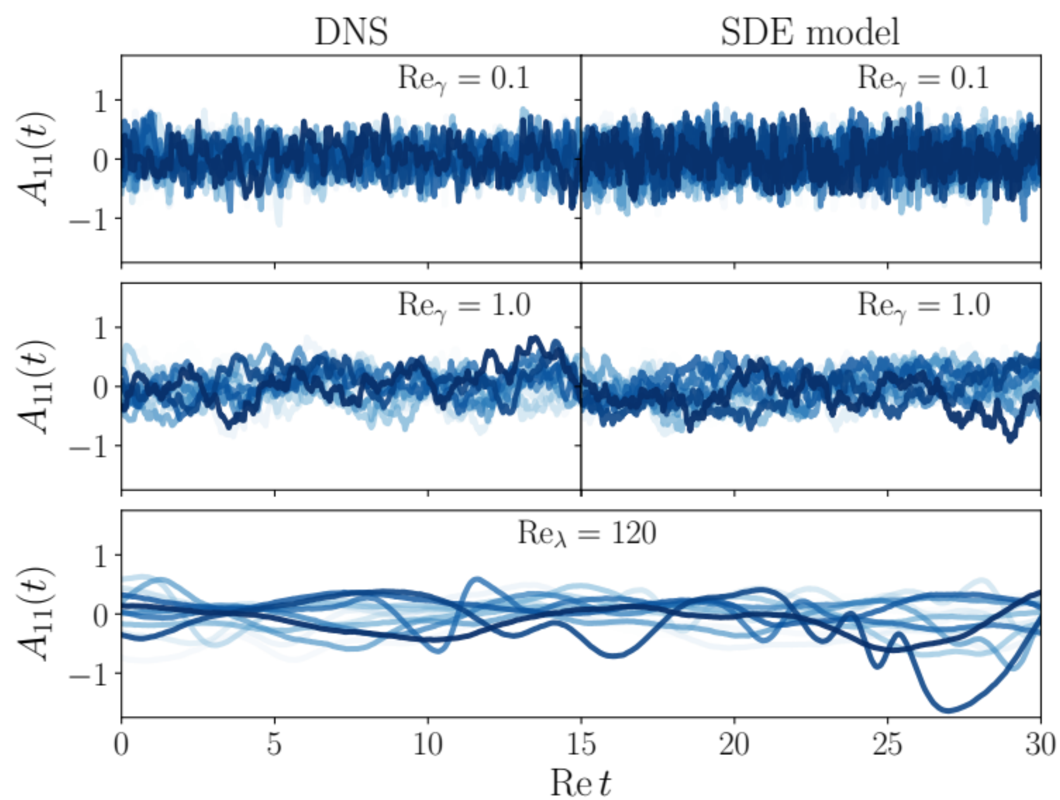
$$C_{\boldsymbol{A}}(\tau) = \frac{\langle A_{ij}(0)A_{ij}(\tau)\rangle}{\langle A_{ij}(0)A_{ij}(0)\rangle}$$



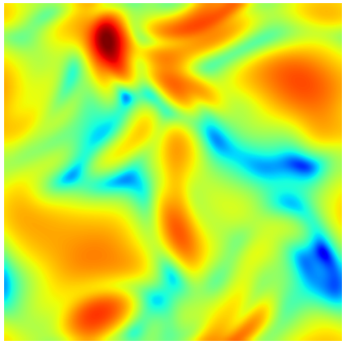
Random flows at low Reynolds: Conclusions



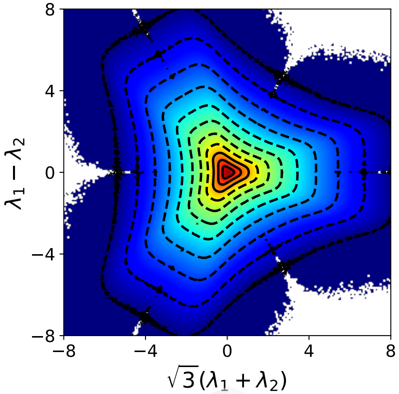
- Closed model for the velocity gradient (no fitting parameters)
- Analytically shown the onset of skewness, alignments, intermittency



- Similar velocity gradient realizations and time correlations in the SDE model and DNS

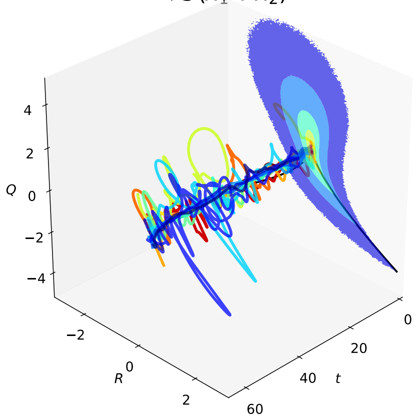


Velocity gradients at low Reynolds numbers

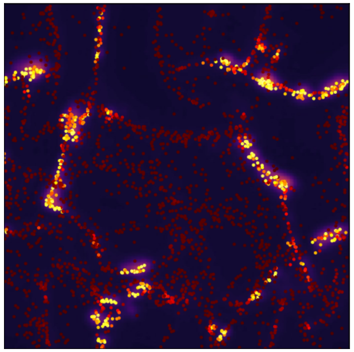


Strain rate at high Reynolds numbers

Acknowledgments: Prof. Michael Wilczek

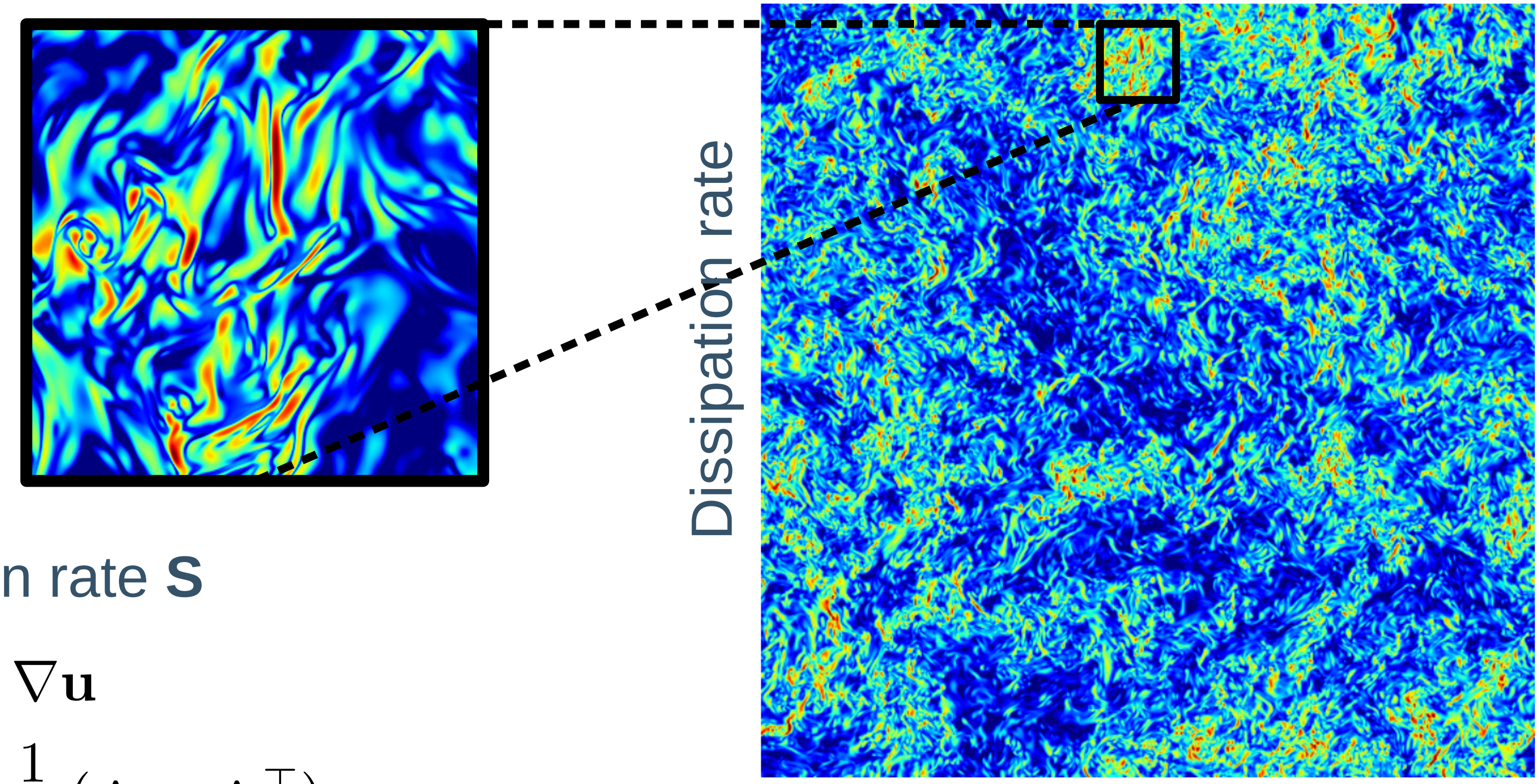


Velocity gradients at high Reynolds numbers



Some applications

Strain rate at high Reynolds



$$\varepsilon = 2\mathrm{Tr}\left(\mathbf{S}^2\right)$$

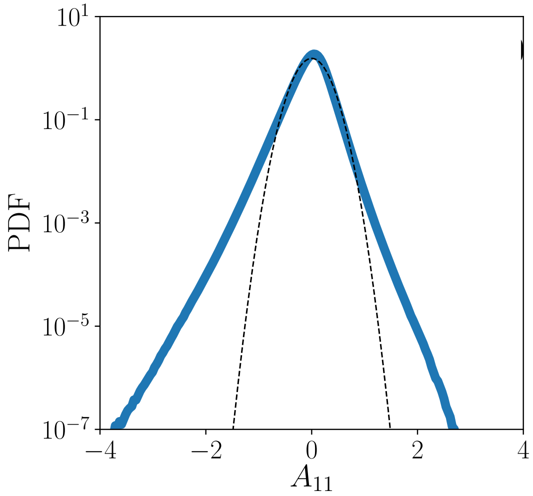
- Strain rate **S**

$$\mathbf{A} = \nabla \mathbf{u}$$

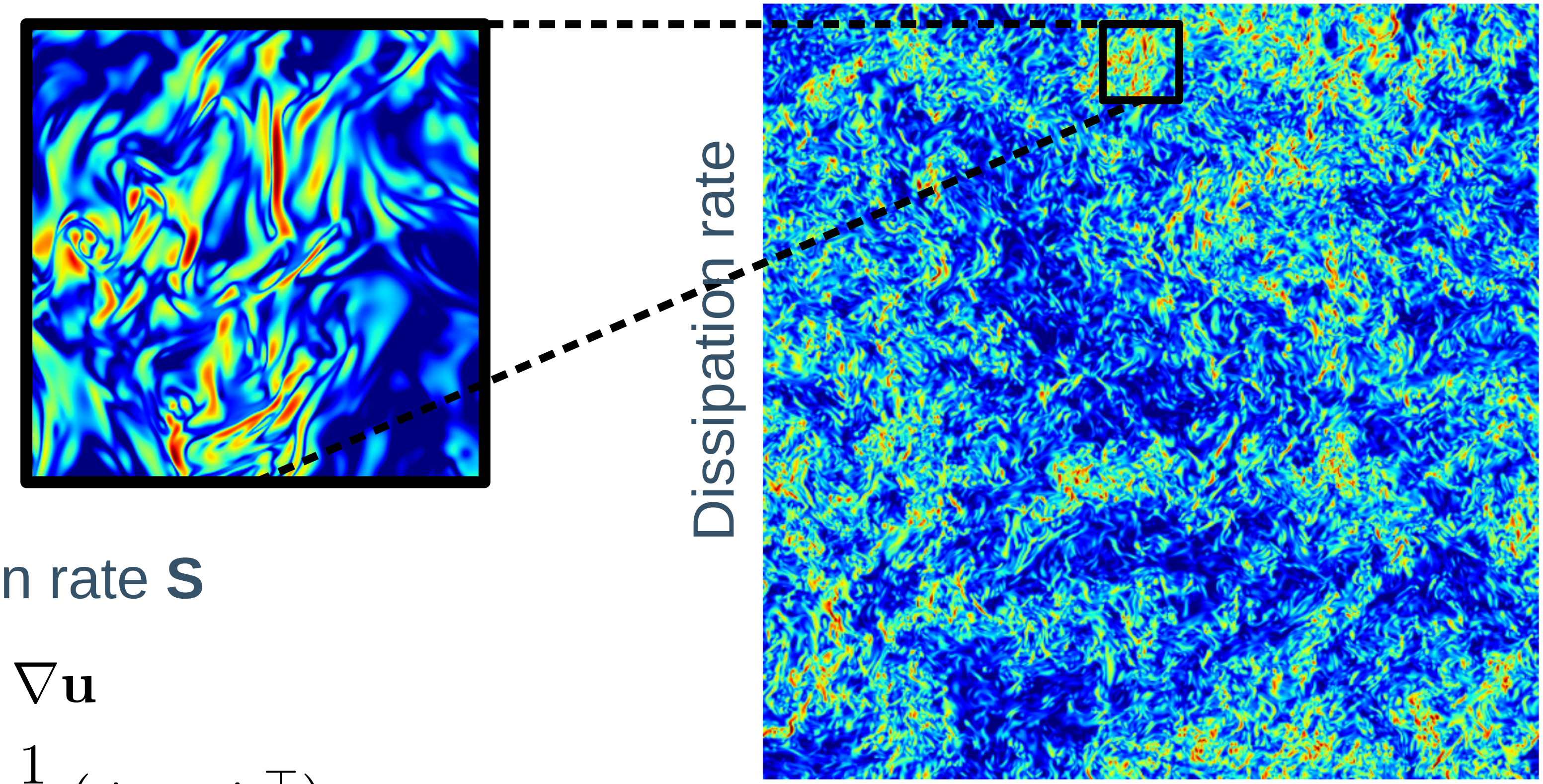
$$\mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^\top\right)$$

- Extreme events: heavy tails

Longitudinal component PDF



Strain rate at high Reynolds



$$\varepsilon = 2\mathrm{Tr}\left(\mathbf{S}^2\right)$$

- Strain rate **S**

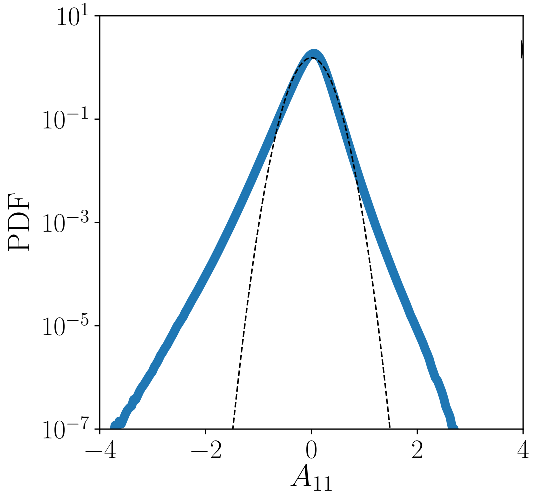
$$\mathbf{A} = \nabla \mathbf{u}$$

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^\top\right)$$

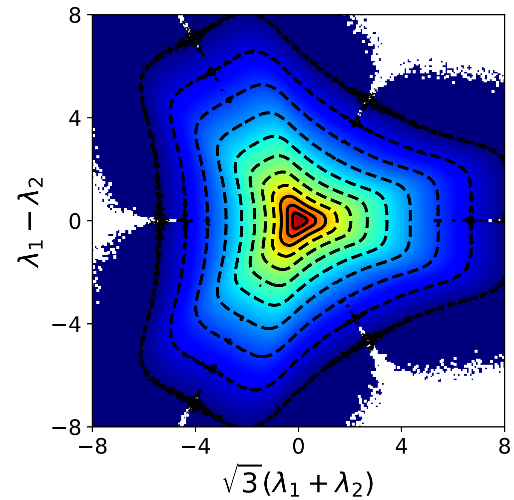
- Extreme events: heavy tails
- Energy cascade: skewness

...Complicated PDFs
...Time correlations

Longitudinal component PDF



Strain-rate PDF



Tailor-made high-Reynolds models

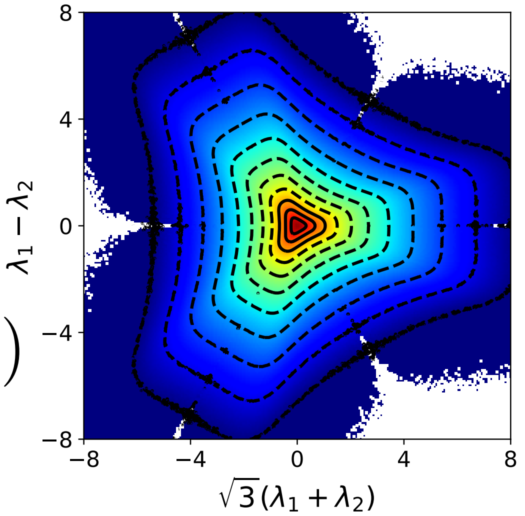
$$\nabla \cdot \mathbf{u} = 0$$

Strain-rate dynamics:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{F}$$

$$\dot{\mathbf{S}} = -\mathbf{S}^2 - \underbrace{\left\langle \mathbf{W}^2 \middle| \mathbf{S} \right\rangle}_{\text{Centrifugal stresses}} - \underbrace{\left\langle \mathbf{H} \middle| \mathbf{S} \right\rangle}_{\text{Pressure Hessian}} + \nu \underbrace{\left\langle \nabla^2 \mathbf{S} \middle| \mathbf{S} \right\rangle}_{\text{Viscous stress}} + \underbrace{\sigma \left(\mathbf{\Gamma} + \mathbf{\Gamma}^\top \right)}_{\text{Tensorial noise}}$$

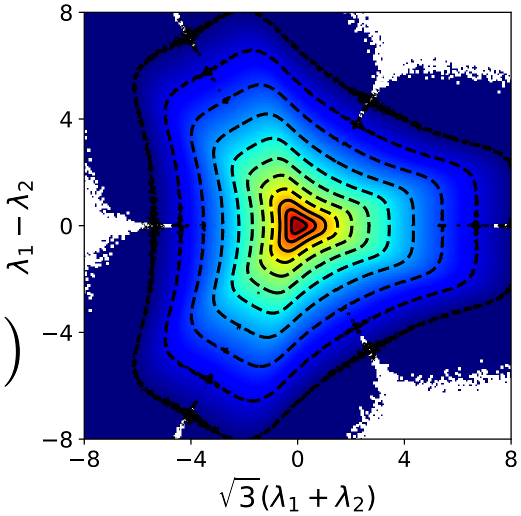
f(x(λ)):



- Single-particle modelling: unclosed equations

Tailor-made high-Reynolds models

$f(x(\lambda))$:



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Centrifugal stresses — Pressure Hessian — Viscous stress — Tensorial noise

$$\frac{\partial}{\partial S_{ij}}\left[\left(-S^2_{ij}+N_{ij}\right)f-\frac{\sigma^2}{2}Q_{ijpq}\frac{\partial f}{\partial S_{pq}}\right]=0$$

- Single-particle modelling: unclosed equations
- Usual: class of models (SDEs), fit the equation parameters to match DNS

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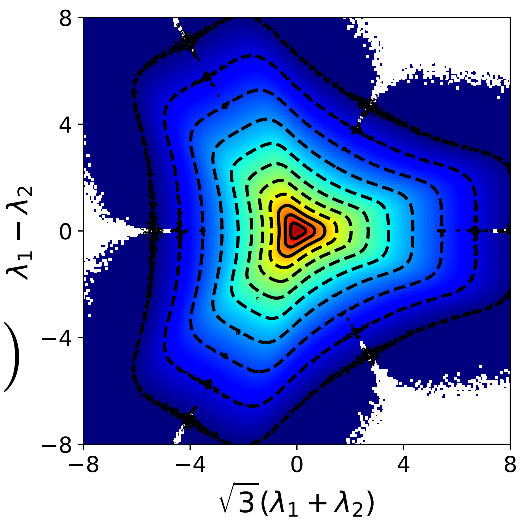

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Centrifugal stresses | Pressure Hessian | Viscous stress | Tensorial noise

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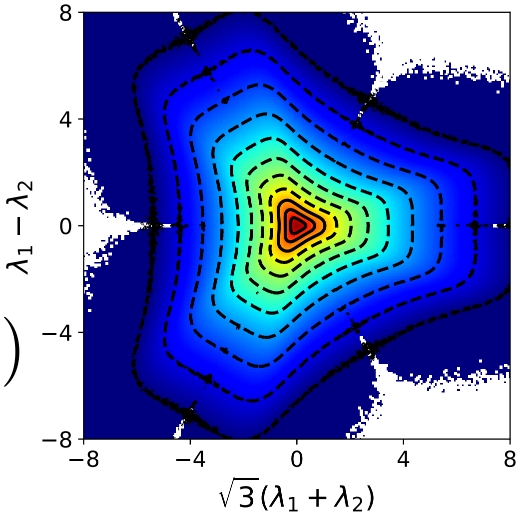

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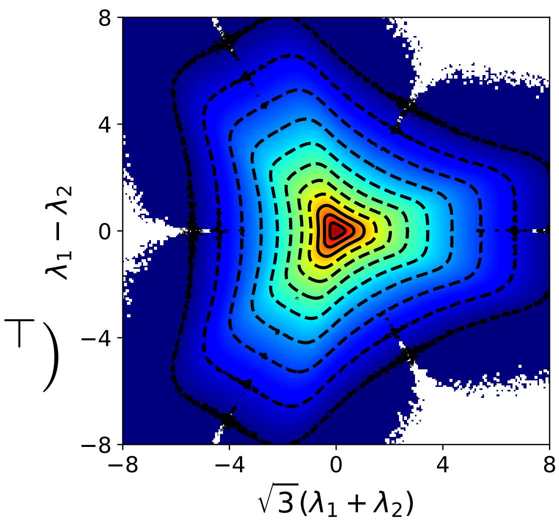
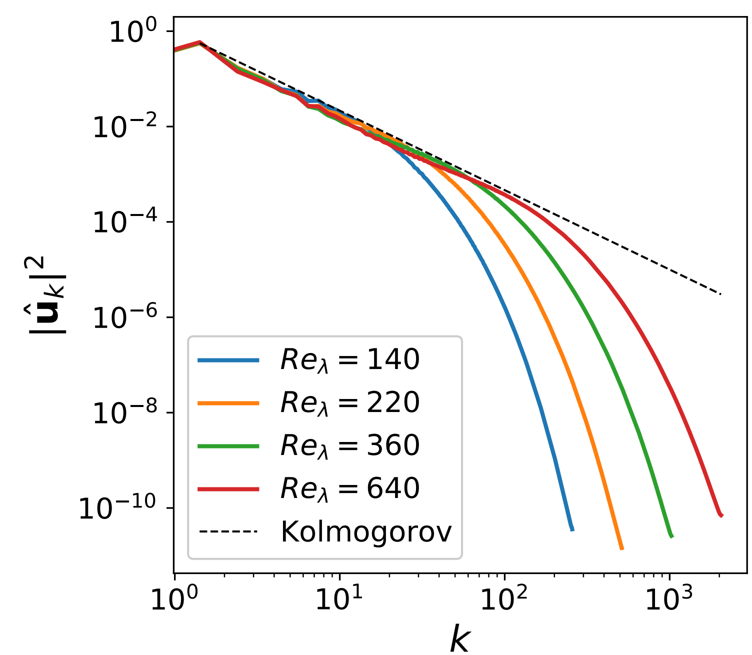
...data-driven, DNS database

f(x(λ)):



DNS: 512^3–4096^3, $k_{max}\eta>3$



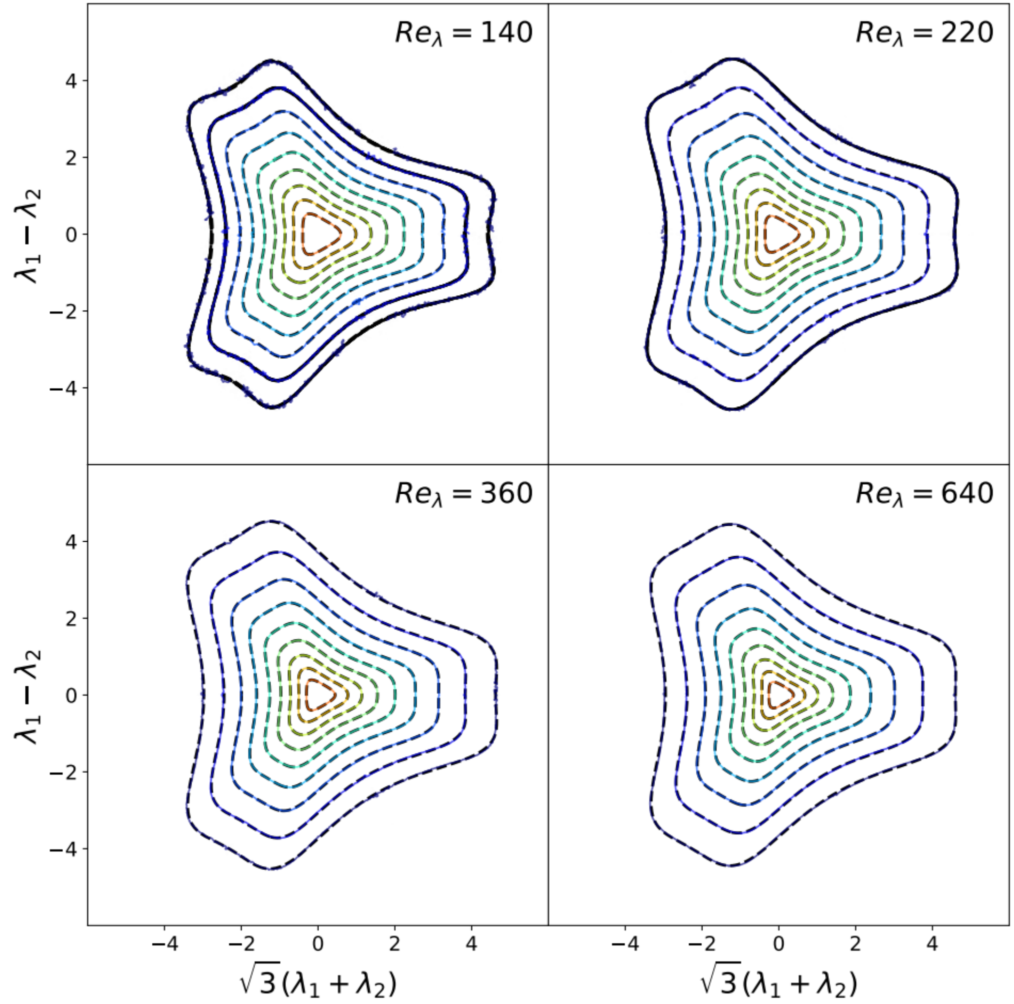
Strain-rate PDF: contours

$$\mathcal{I}_1 = \mathrm{Tr}\left(\mathbf{S}^2\right),\ \mathcal{I}_3 = \mathrm{Tr}\left(\mathbf{S}^3\right)$$

- Contours can be fitted ≈ exactly

$$\alpha_0(f) = \mathcal{I}_1^3 + \alpha_1(f)\mathcal{I}_1^{3/2}\mathcal{I}_3 + \alpha_2(f)\mathcal{I}_3^2$$

- α_1: skewness



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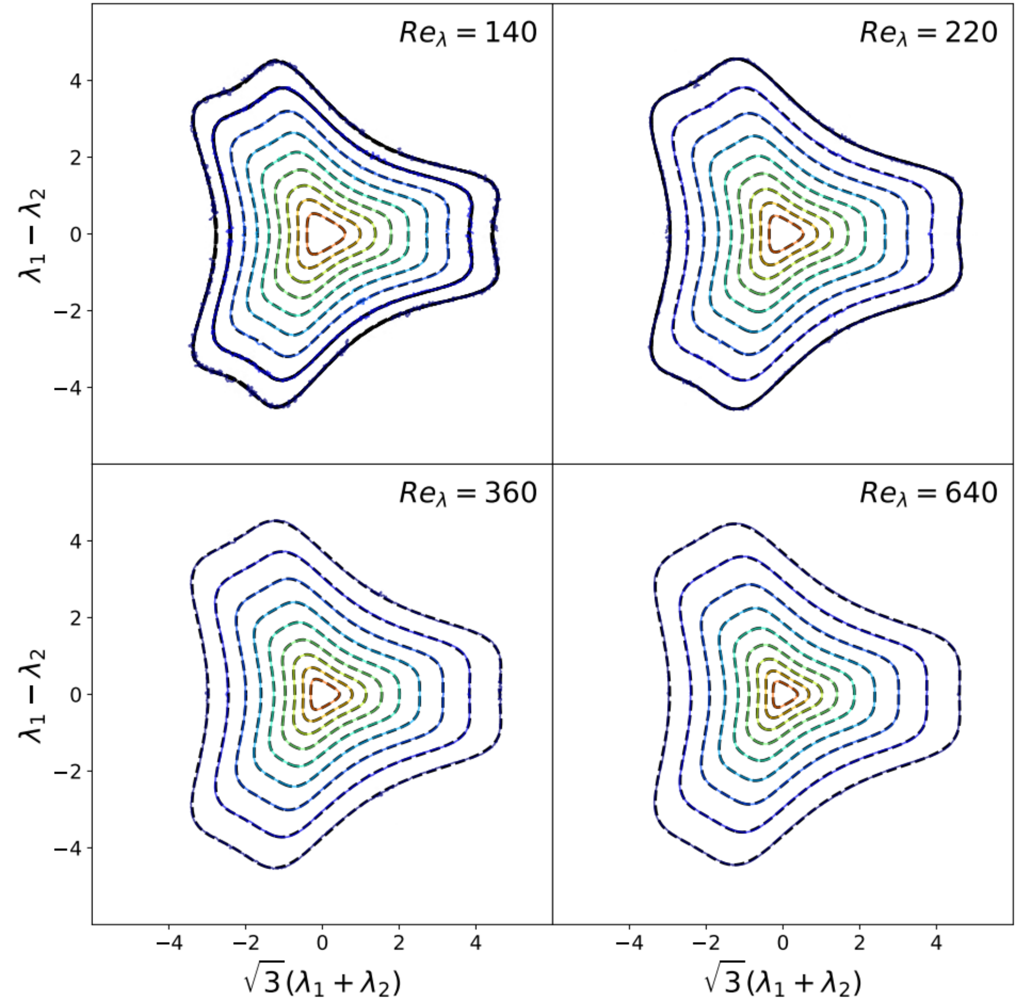
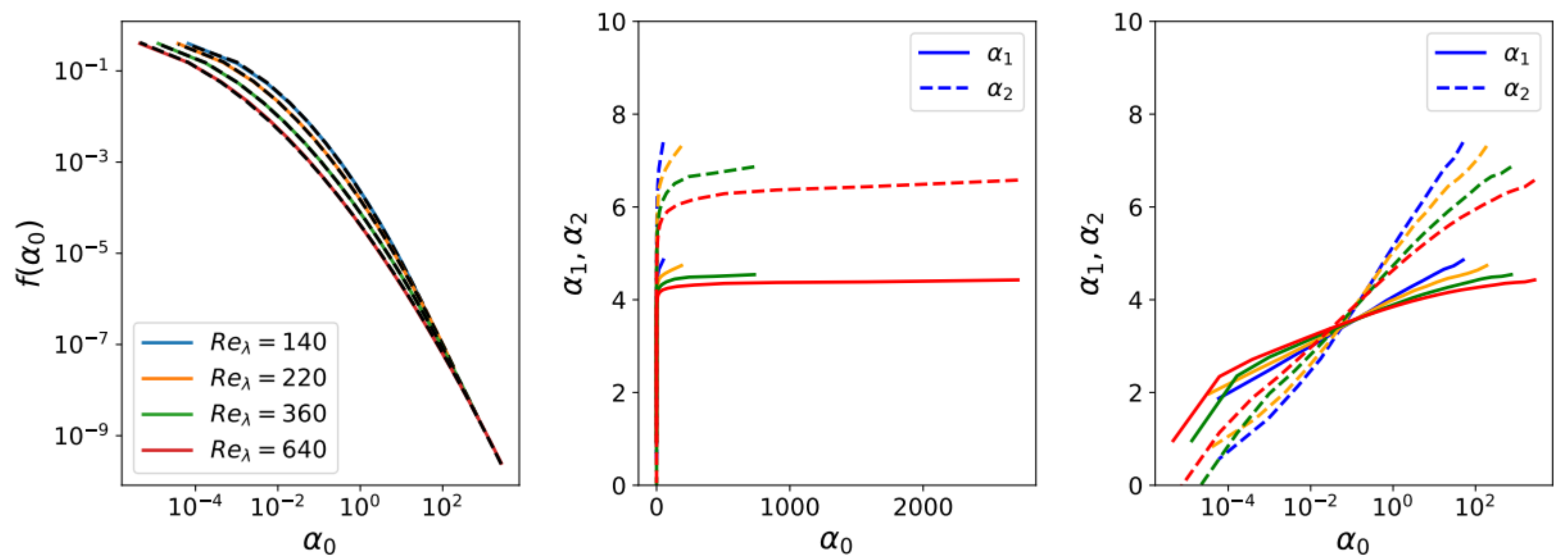
- α_1: skewness
- Approximations on the coefficients
- *f:* lognormal across contours





Strain-rate PDF: whole PDF

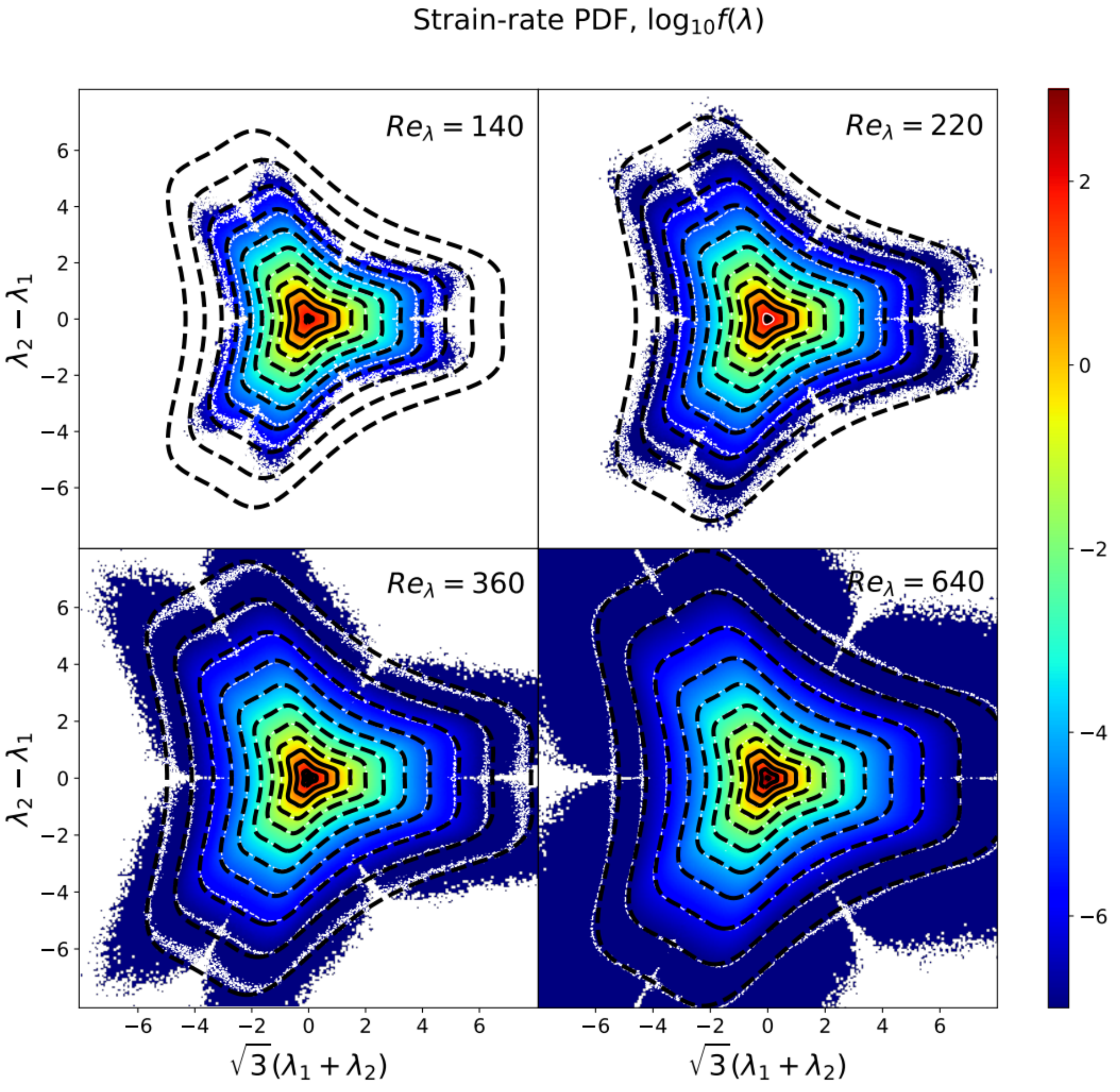
$$\mathcal{I}_1 = \mathrm{Tr}\left(\mathbf{S}^2\right), \; \mathcal{I}_3 = \mathrm{Tr}\left(\mathbf{S}^3\right)$$

Fitting goes like..

$$\alpha_0 = \mathcal{I}_1^3 + \alpha_1 \mathcal{I}_1^{3/2} \mathcal{I}_3 + \alpha_2 \mathcal{I}_3^2$$

$$f(\mathbf{S}) \approx f(\alpha_0)$$

$$f = \mathcal{N} \exp\left(- \frac{(\log \alpha_0 - \mu)^2}{\Sigma^2} \right)$$



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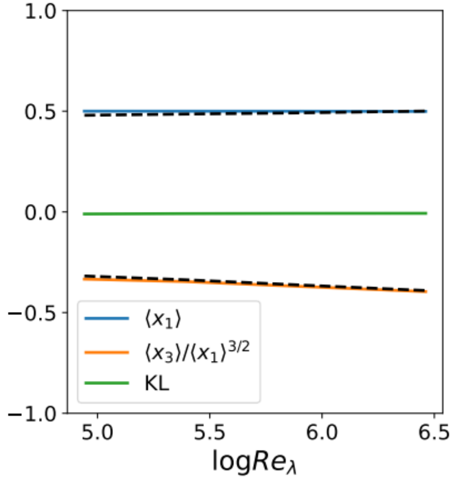
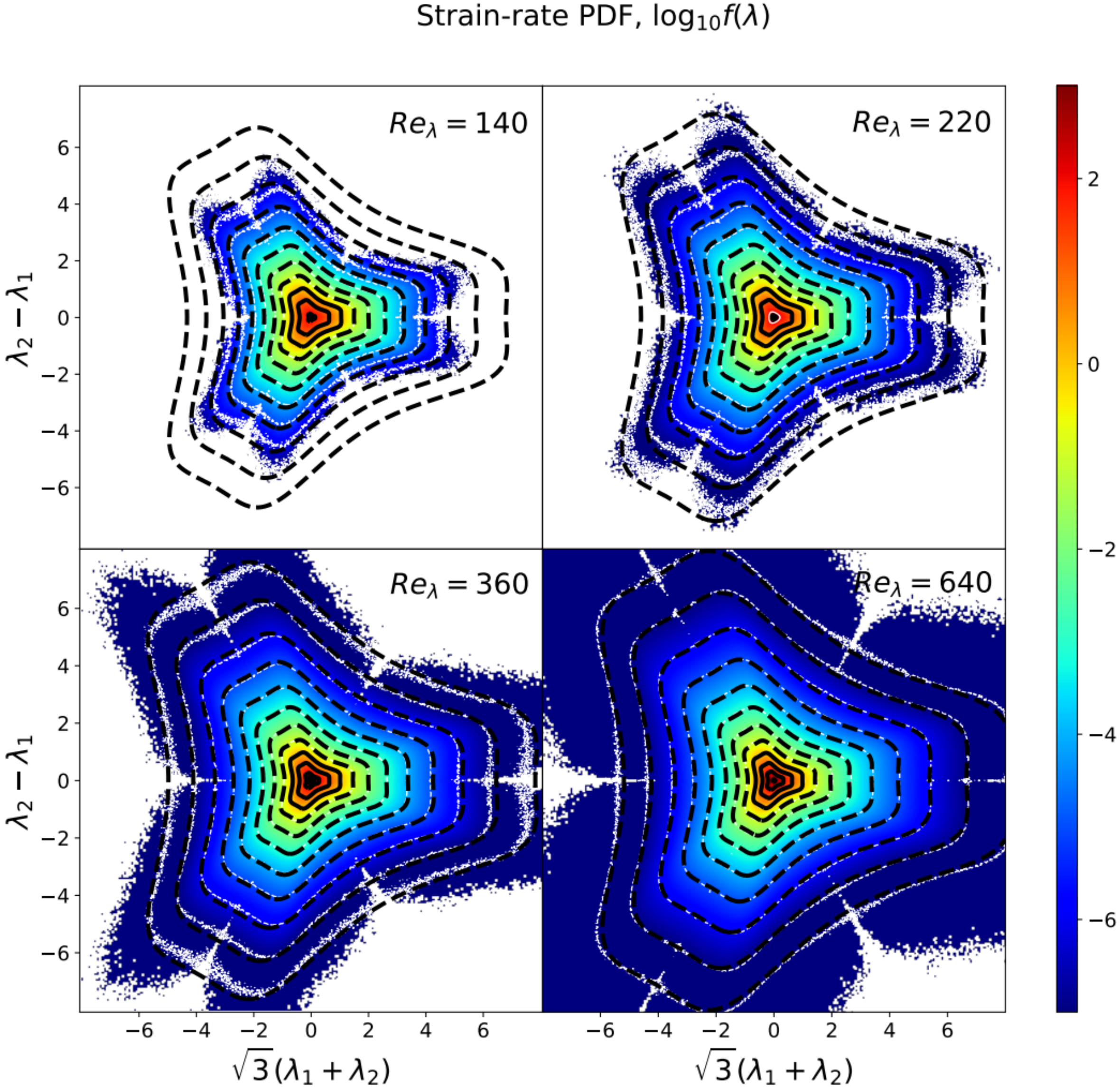
$$\alpha_0 = \mathcal{I}_1^3 + \alpha_1 \mathcal{I}_1^{3/2}\mathcal{I}_3 + \alpha_2 \mathcal{I}_3^2$$

$$f(\mathbf{S}) \approx f(\alpha_0)$$

$$f = \mathcal{N} \exp\left(-\frac{(\log \alpha_0 - \mu)^2}{\Sigma^2}\right)$$





- Capture PDF moments (core) and tails
- Minimal number of parameters

Strain-rate PDF: whole PDF

$$\mathcal{I}_1 = \mathrm{Tr}\left(\mathbf{S}^2\right),\ \mathcal{I}_3 = \mathrm{Tr}\left(\mathbf{S}^3\right)$$

Fitting goes like..

$$\alpha_0 = \mathcal{I}_1^3 + \alpha_1 \mathcal{I}_1^{3/2}\mathcal{I}_3 + \alpha_2 \mathcal{I}_3^2$$

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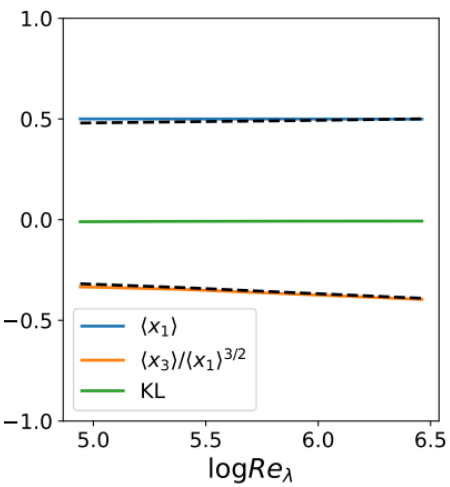
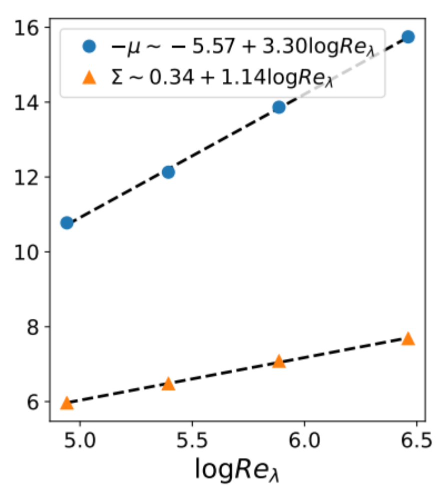
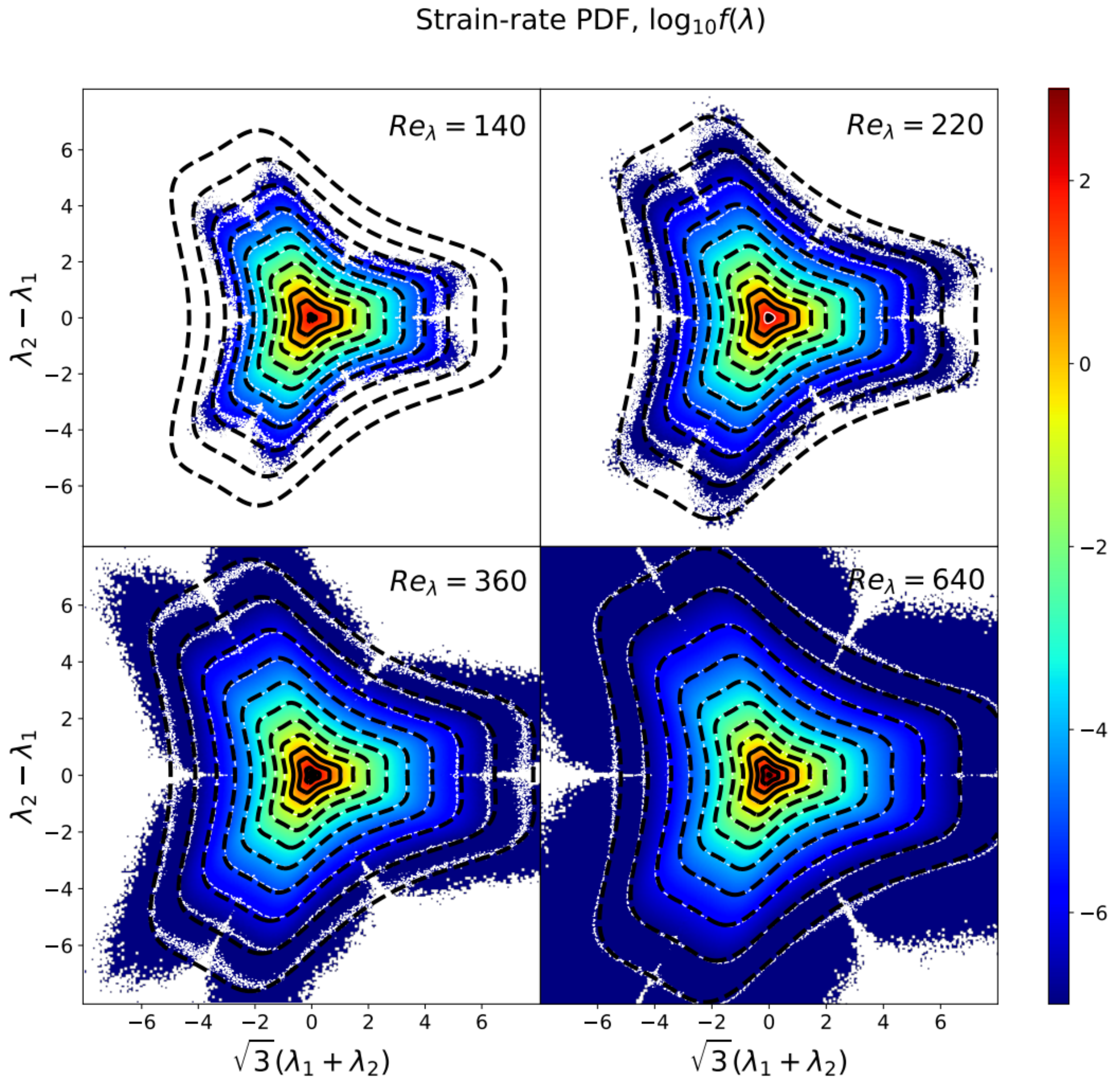
Role of the Reynolds number

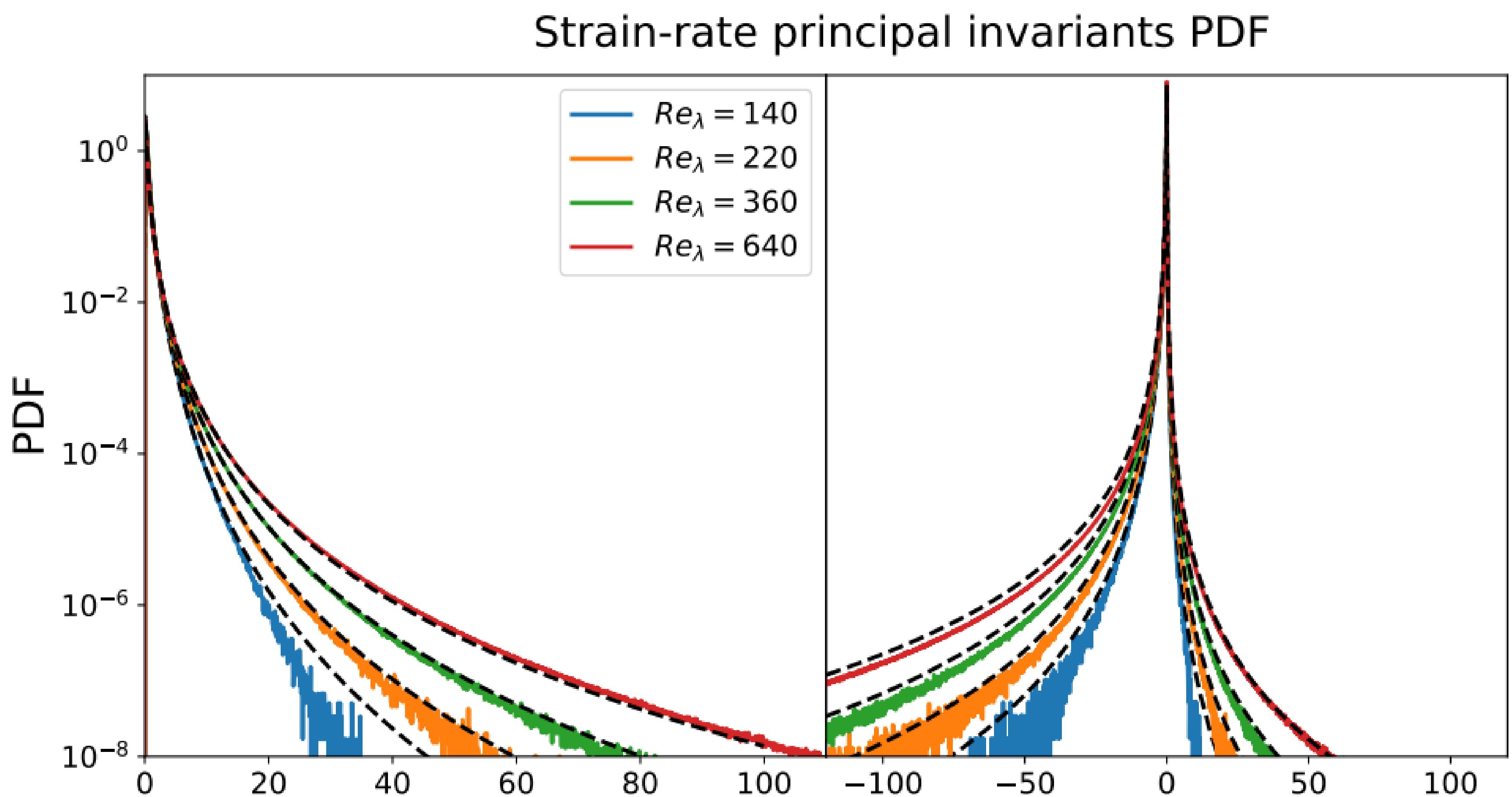






- Capture PDF moments (core) and tails
- Minimal number of parameters
- Extrapolate trends with Re$_\lambda$

Marginals



dissipation rate $\mathcal{I}_1 = \mathrm{Tr}\left(\mathbf{S}^2\right)$

strain self amplification $\mathcal{I}_3 = \mathrm{Tr}\left(\mathbf{S}^3\right)$

The geometry of 3x3 symmetric, traceless, isotropic matrix:

$$\mathrm{d}\mathbf{S} f(\mathbf{S}) = \mathcal{N} \mathrm{d}\mathcal{I}_1 \mathrm{d}\mathcal{I}_3 f(\mathcal{I}_1, \mathcal{I}_3) = \mathcal{N}' \Pi_{i \neq j} |\lambda_i - \lambda_j| \mathrm{d}\lambda_i f(\lambda)$$

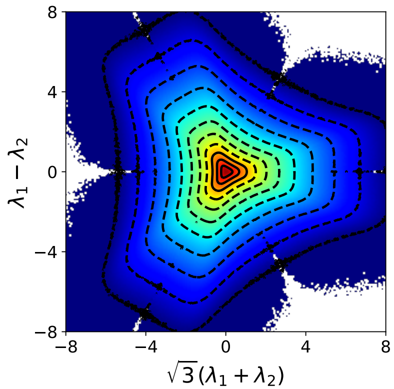
Cartesian — Traces — Eigenframe

Tailor-made Langevin and FP equation

So far:

- Analytic PDF, no need to run simulations!
- Time correlations?

f(x(λ)):



Fokker-Planck equation + tensor function representation

Unclosed Known (fitting)

$$\frac{\partial}{\partial S_{ij}}\left[\left(-S_{ij}^2 + \boxed{N_{ij}}\right)\boxed{f} - \frac{\sigma^2}{2} Q_{ijpq} \frac{\partial f}{\partial S_{pq}}\right] = 0$$

$$N_{ij} = \text{centrifugal} + \text{pressure Hessian} + \text{viscous stresses} = \sum_{n=1}^{d} \boxed{\gamma_n(\mathcal{I})} B_{ij}^n$$

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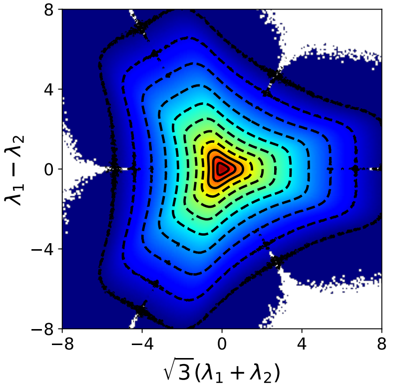

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Momentarily assume detailed balance: get coefficients

$$-S_{ij}^2+\gamma_n B_{ij}^n=\frac{\sigma^2}{2}Q_{ijpq}\frac{\partial\log f}{\partial S_{pq}}\longrightarrow\gamma_n(\mathcal{I})$$

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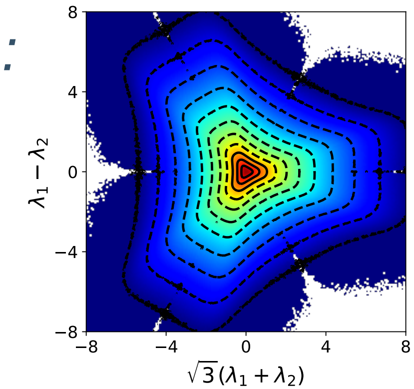

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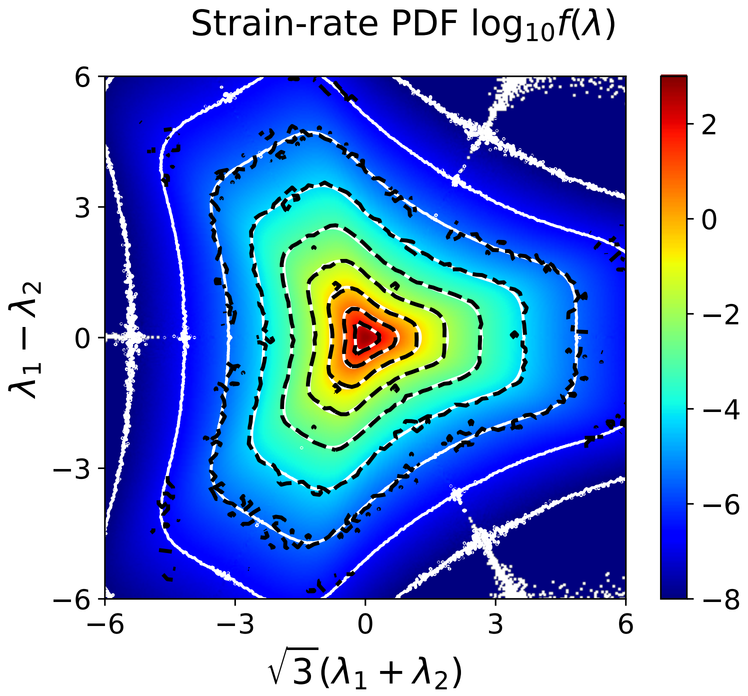
Tools

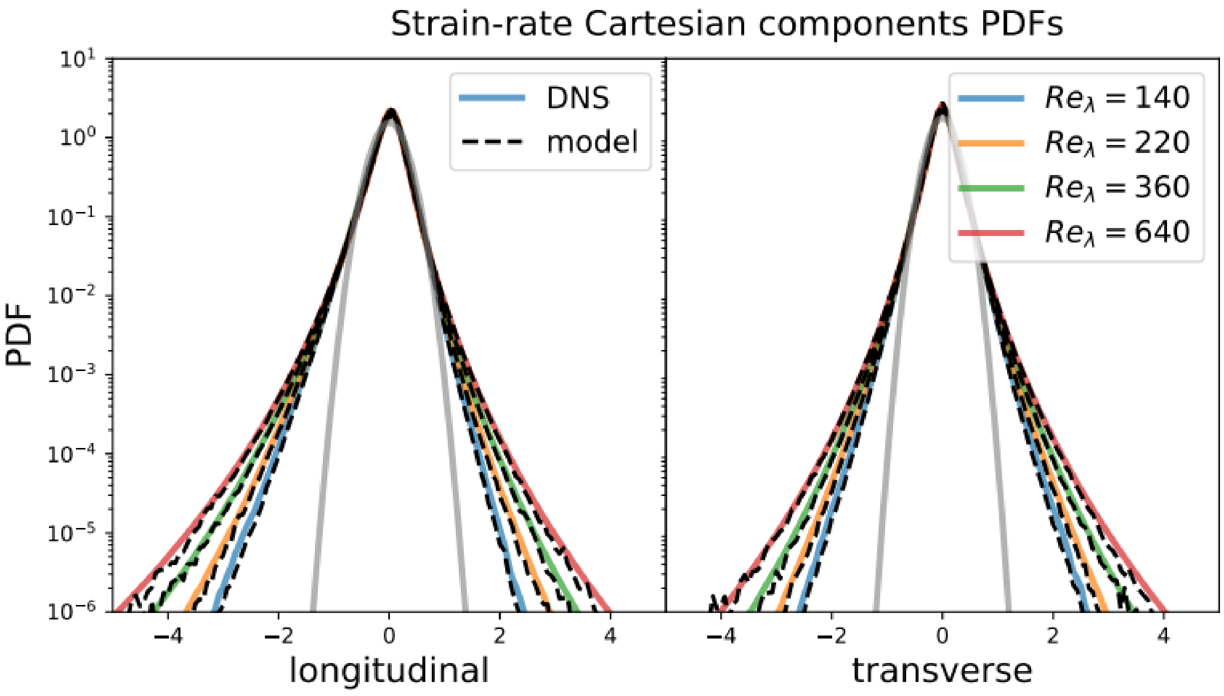
$$\frac{\partial f}{\partial S_{pq}}(\mathcal{I})=\frac{\partial f}{\partial \mathcal{I}_k}\overset{[1]}{M_{kn}}\boxed{B^n_{pq}}$$ Basis tensors from ***S***

$$\frac{\partial B^n_{ij}}{\partial S_{pq}}=\Gamma^{n,0}_{lm}B^l_{ij}B^m_{pq}+\overset{[1]}{\Gamma^{n,1}_{lm}}B^l_{ip}B^m_{jq}+\Gamma^{n,2}_{lm}B^l_{iq}B^m_{jp}$$

[1] Carbone and Wilczek, JFM **948**, (2022)

Strain-rate dynamics from Langevin eq





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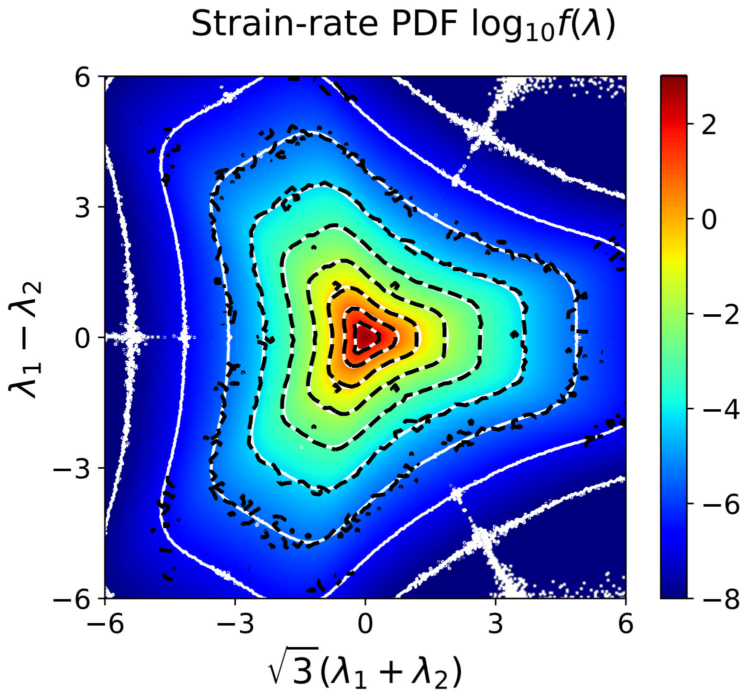
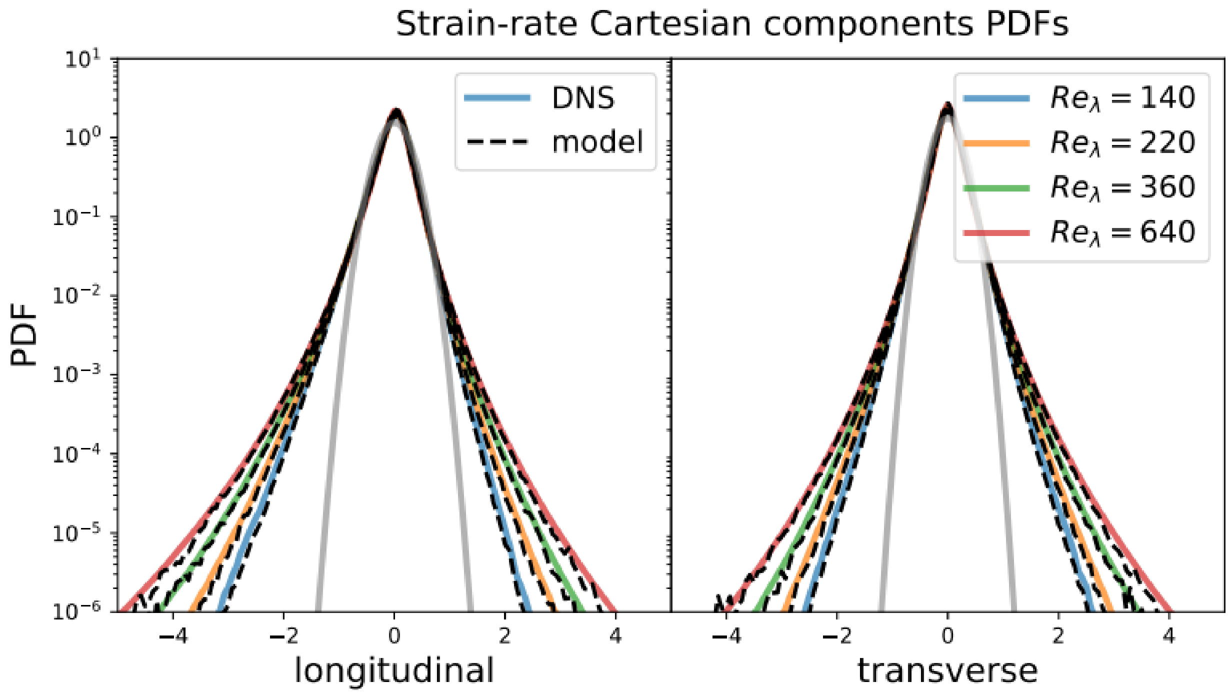




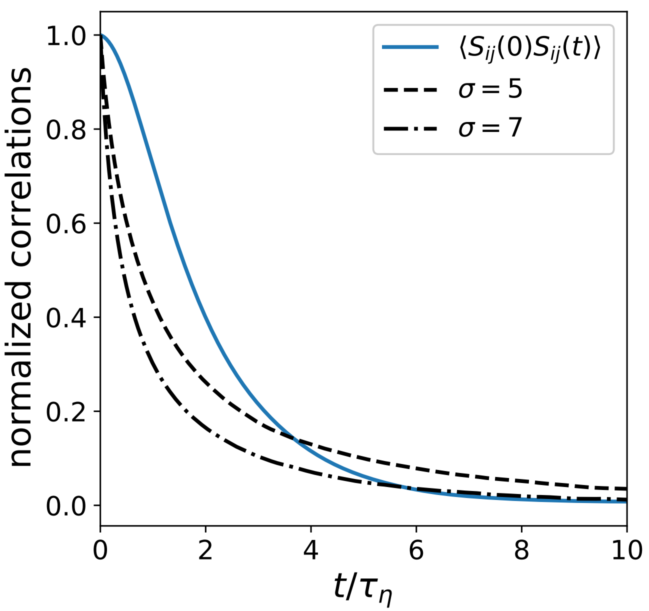
Beyond detailed balance:

- Gauge terms for single-time stats

$$\frac{\partial}{\partial S_{ij}}(T_{ij} f) = 0$$ leaves FPE unchanged

- Simple symmetries of the FPE

$$\sigma \to \sigma'$$



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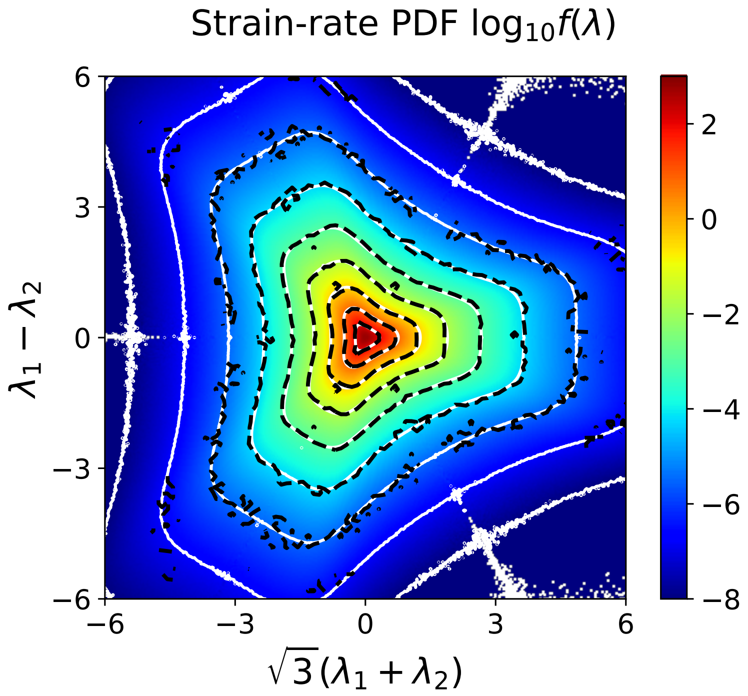
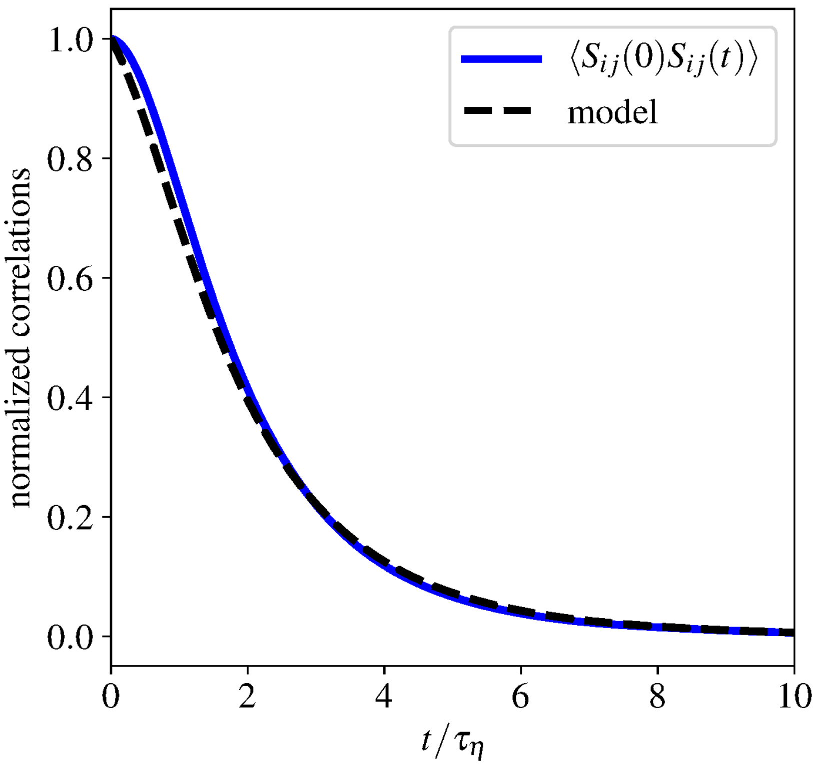


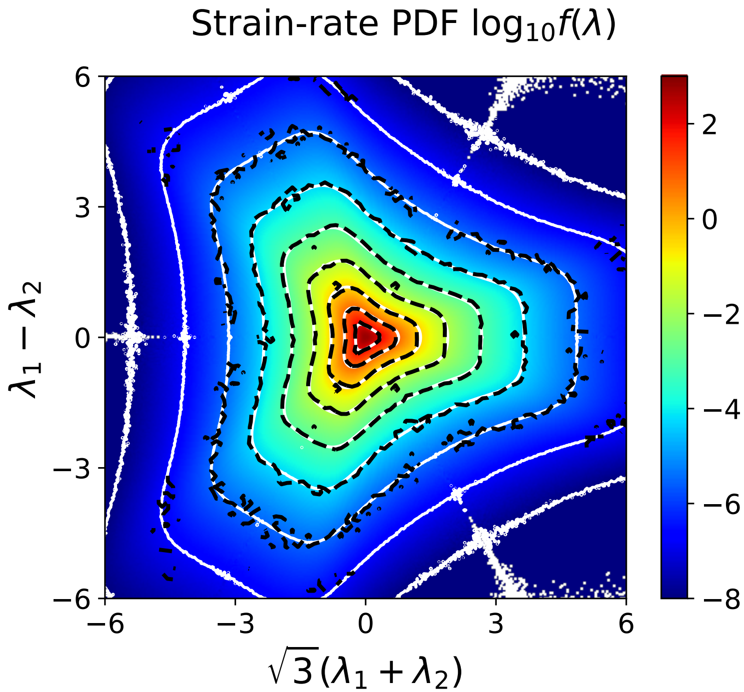


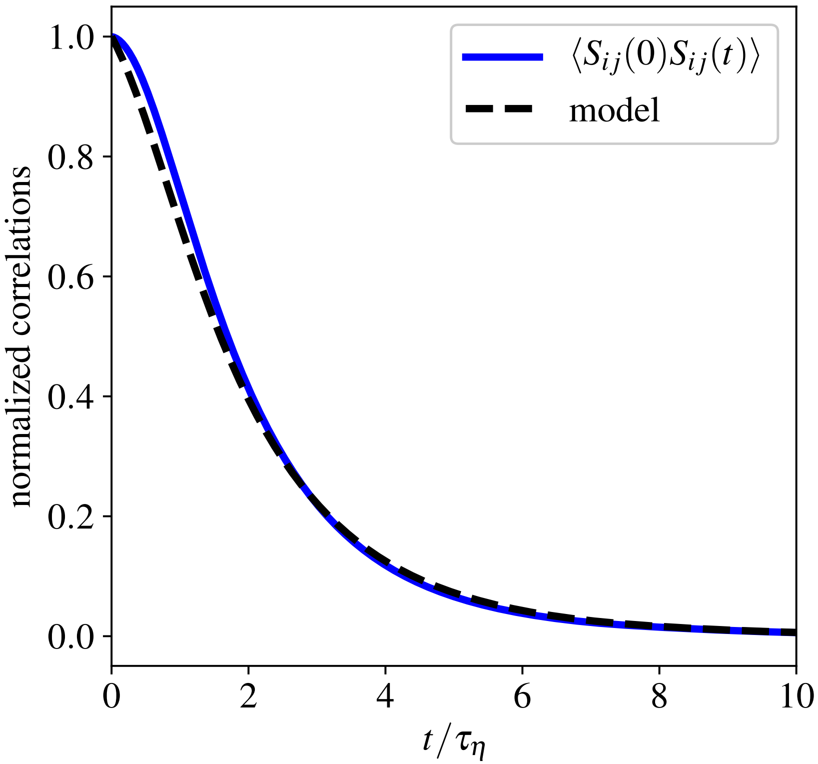
Beyond detailed balance:

- Multiplicative noise (eigenframe rotation)

$$d\boldsymbol{S} = \boldsymbol{N}\mathrm{d}t + \sqrt{2\mathrm{d}t}\left[\sigma\boldsymbol{\Gamma} + g\left(\boldsymbol{S}\boldsymbol{\mathcal{W}} - \boldsymbol{\mathcal{W}}\boldsymbol{S}\right)\right]$$

Strain-rate dynamics from Langevin eq





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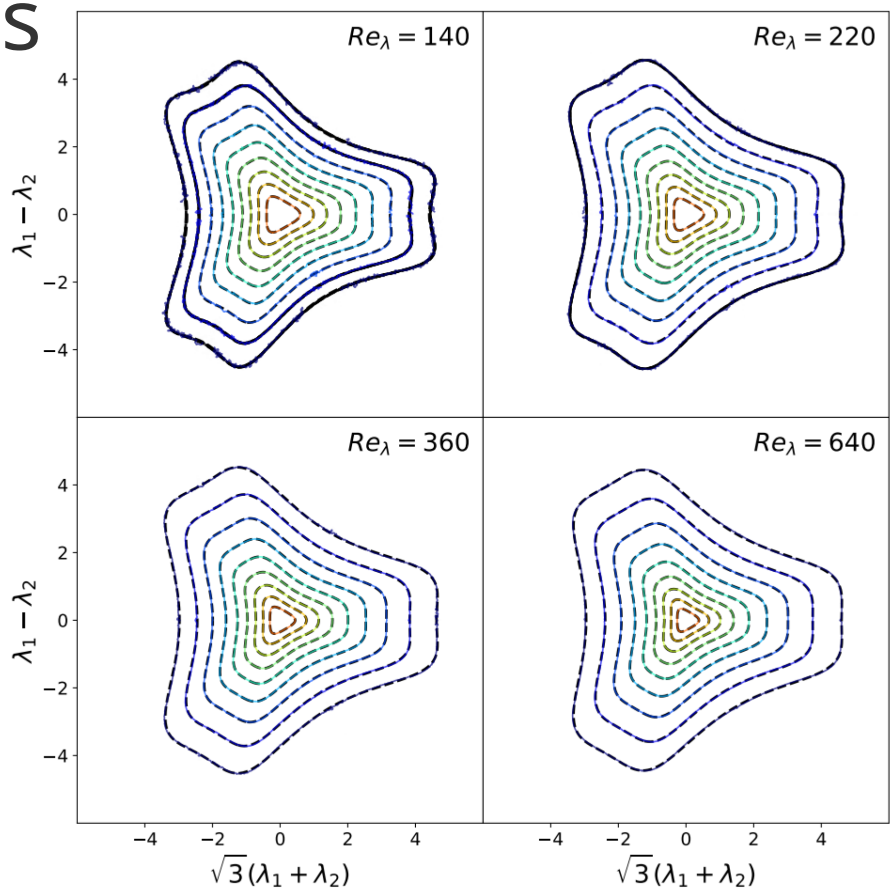
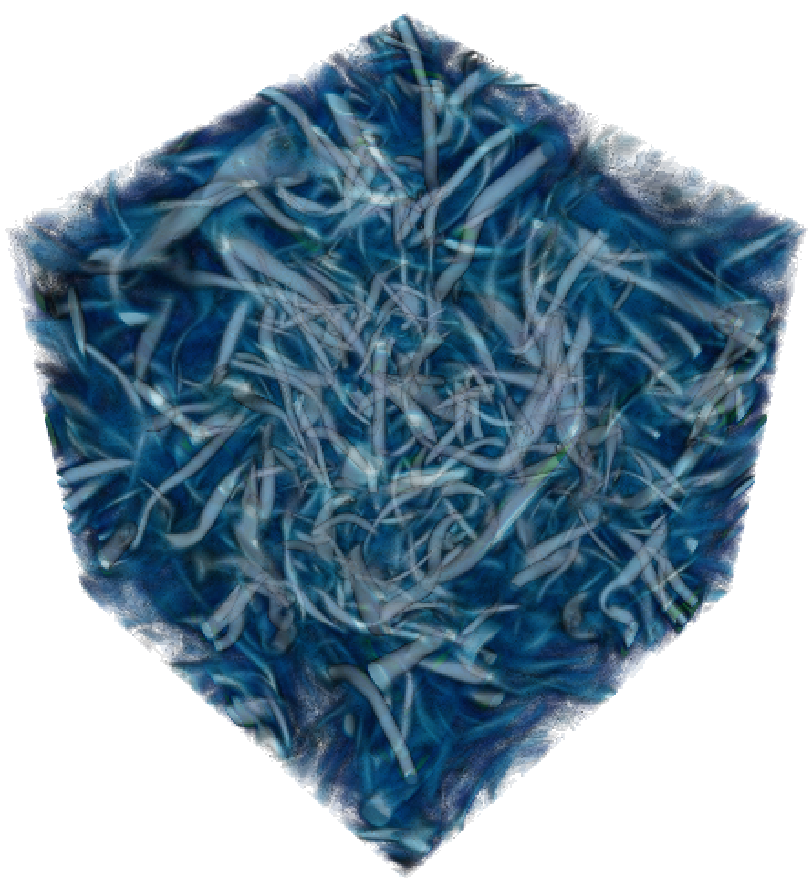
Symmetric Anti-symmetric
Gaussian white noise

Carbone, Iovieno, Bragg, "Symmetry transformation and dimensionality reduction of the anisotropic pressure Hessian", *JFM* **900**, (2020)

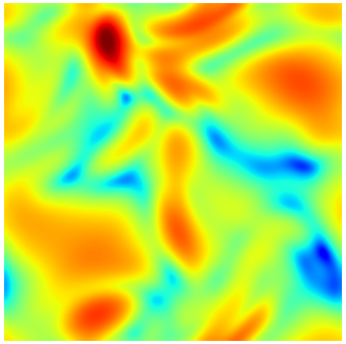
Phenomenological modeling at high Reynolds: Conclusions



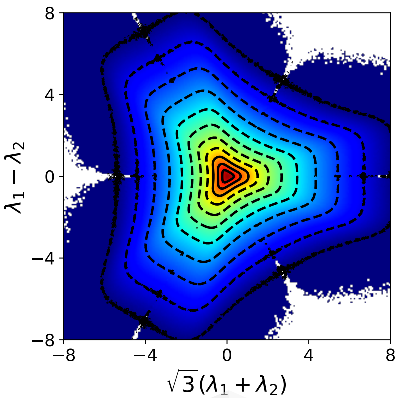
- Model designed for the strain-rate ..single-point stats not so complicated



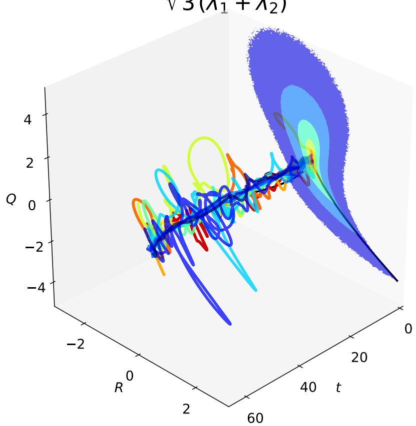
- Why that contours shape?
- Extend the fitting to the full gradient PDF (5D)



Velocity gradients at low Reynolds numbers

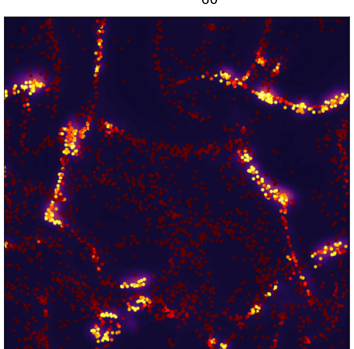


Strain rate at high Reynolds numbers



<u>Velocity gradients at high Reynolds numbers</u>

Acknowledgments: Vincent Peterhans, Prof. Alexander Ecker, Prof. Michael Wilczek



Some applications

A data-driven model for the small scales?

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{F}$$

- Single-particle, Lagrangian viewpoint
- Trajectories from Navier-Stokes: **non-local**

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Velocity gradient dynamics:

$$\underbrace{\mathrm{d}_t \mathbf{A}}_{\text{Lagrangian derivative}} = -\mathbf{A}^2 - \overbrace{\mathbf{H}}^{\text{Pressure Hessian}} + \underbrace{\nu \nabla^2 \mathbf{A}}_{\text{Viscous stress}} + \overbrace{\nabla \mathbf{F}}^{\text{External forcing}}$$

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- Model: Langevin equation, local up to noise

Drift

Gaussian white noise

$$\boxed{d\boldsymbol{\mathcal{A}} = \boldsymbol{N}(\boldsymbol{\mathcal{A}}, \ldots)\mathrm{d}t + \sqrt{2\mathrm{d}t}\,\boldsymbol{\Sigma}(\boldsymbol{\mathcal{A}}, \ldots)\boldsymbol{\Gamma}}$$

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- $\boldsymbol{A} \simeq \boldsymbol{\mathcal{A}}$: model and DNS trajectories *statistically* similar

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- $\boldsymbol{A} \simeq \boldsymbol{\mathcal{A}}$: model and DNS trajectories *statistically* similar
- How? Learn the PDF of ***A***..

  ..construct a model featuring that steady-state PDF

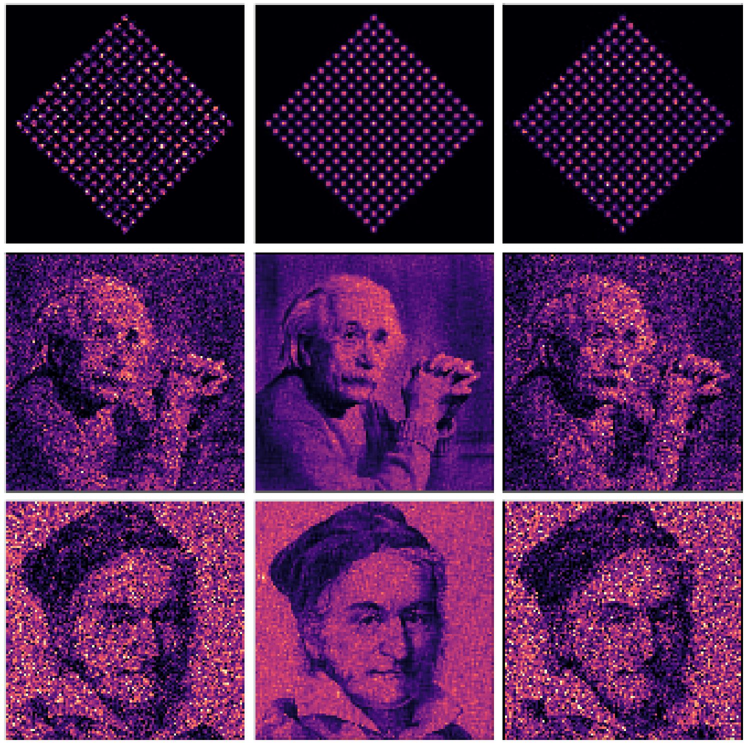
Normalizing flow for tailor-designed models

How can we learn a PDF?

- Transform Gaussian into a target PDF
  [E. G. Tabak, E. Vanden-Eijnden, (2010)]

- Not just one shot.. Sequence of simple invertible transformations
  [L. Dinh, J. Sohl-Dickstein, S. Bengio, (2017)]

- Works with high-dimensional PDFs (images)



Durkan, Bekasov, Murray and Papamakarios, “Cubic-Spline Flows”, arXiv:1906.02145 [stat.ML], 2019

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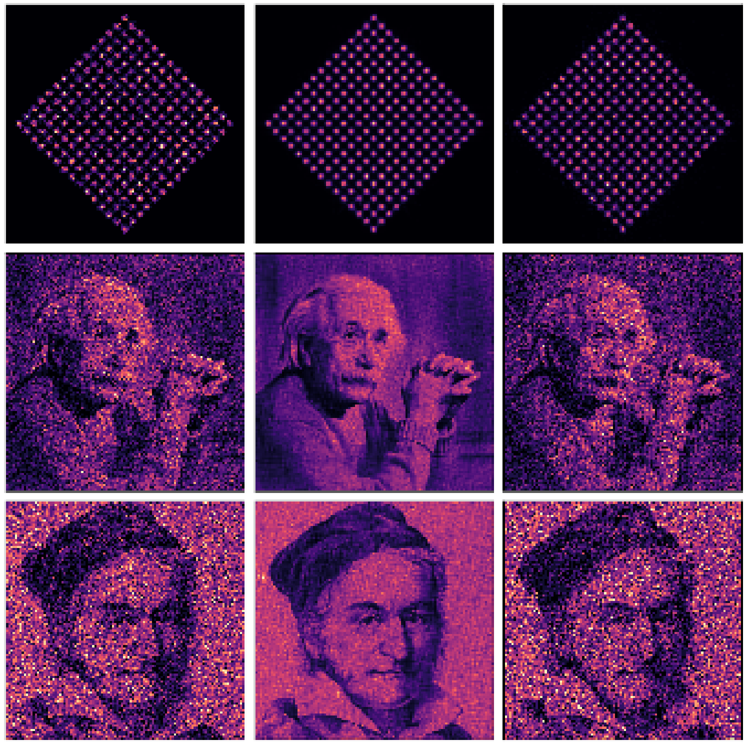

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Gaussian random matrices → turbulent-like ensemble

Learning the velocity gradient PDF

$\boldsymbol{\mathcal{A}}^{(0)} \sim g$ $\boldsymbol{\mathcal{A}}^{(32)} \sim f$

- Gaussian … ***A***$^{(K)}$ *layer K* ***A***$^{(K+1)}$ … Turbulent

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Jacobian of K-*th* transformation

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f(***A***): PDF of the turbulent velocity gradients

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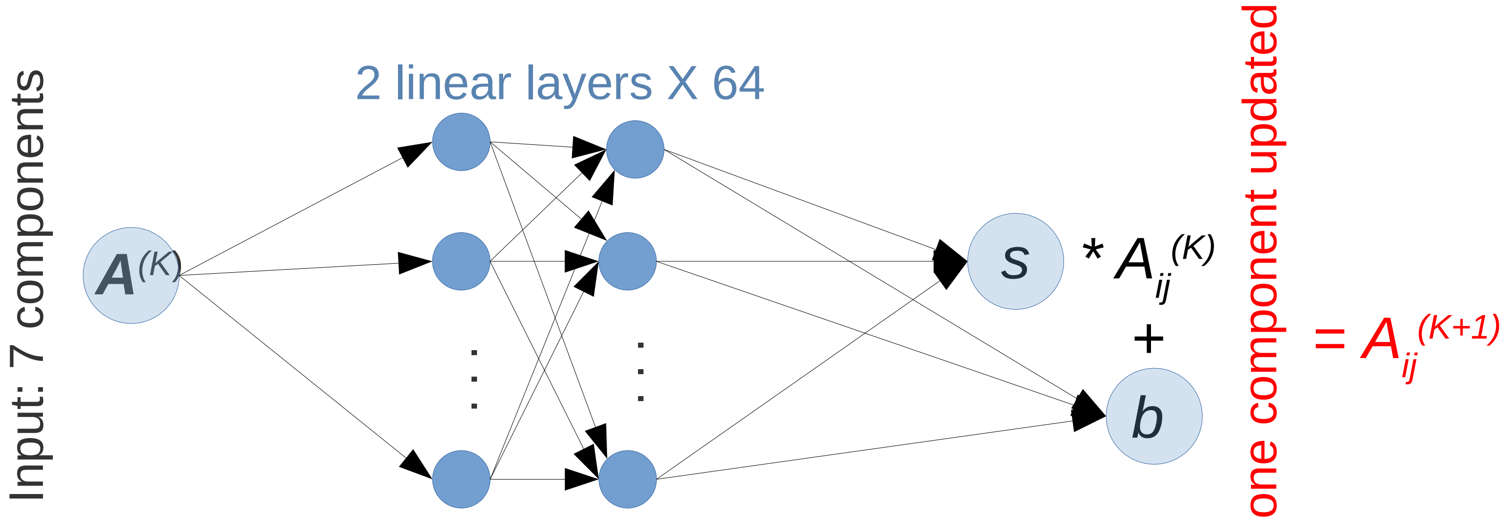

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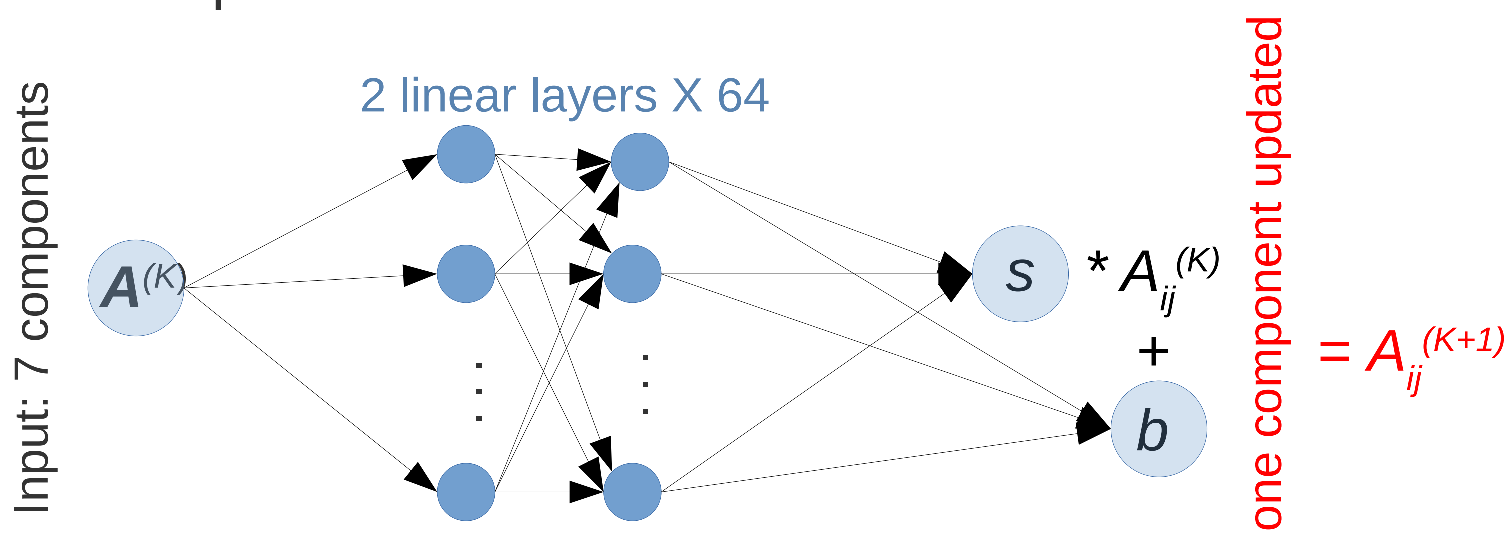

- Maximum likelihood of the turbulent velocity gradient ensemble

$$\max_{\theta} \langle \log f(\mathbf{A}; \theta) \rangle$$

Enforcing single-time statistics in our model

- Learned *f(**A**)* through normalizing flow

*f(**A**)* = PDF of turbulent velocity gradients

- Reduced-order model..

$$\mathrm{d}_t \boldsymbol{\mathcal{A}} = \boldsymbol{N}(\boldsymbol{\mathcal{A}})$$

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..Liouville equation for single-time PDF

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..Liouville equation for single-time PDF

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathcal{A}_{ij}}(N_{ij} f) = 0$$

Learned

- Drift such that learned *f(**A**)* is a steady-state solution

$$N_{ij}(\boldsymbol{\mathcal{A}}; \psi) = \frac{\partial T_{ijpq}}{\partial \mathcal{A}_{pq}} + T_{ijpq} \frac{\partial \log f}{\partial \mathcal{A}_{pq}}$$

$$T_{ijpq} = -T_{pqij}$$

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$$\mathrm{d}_t \boldsymbol{\mathcal{A}} = \boldsymbol{N}(\boldsymbol{\mathcal{A}})$$

..Liouville equation for single-time PDF

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathcal{A}_{ij}}(N_{ij} f) = 0$$

Learned

- Drift such that learned *f(**A**)* is a steady-state solution

$$N_{ij}(\boldsymbol{\mathcal{A}};\psi) = \frac{\partial T_{ijpq}}{\partial \mathcal{A}_{pq}} + T_{ijpq}\frac{\partial \log f}{\partial \mathcal{A}_{pq}}$$

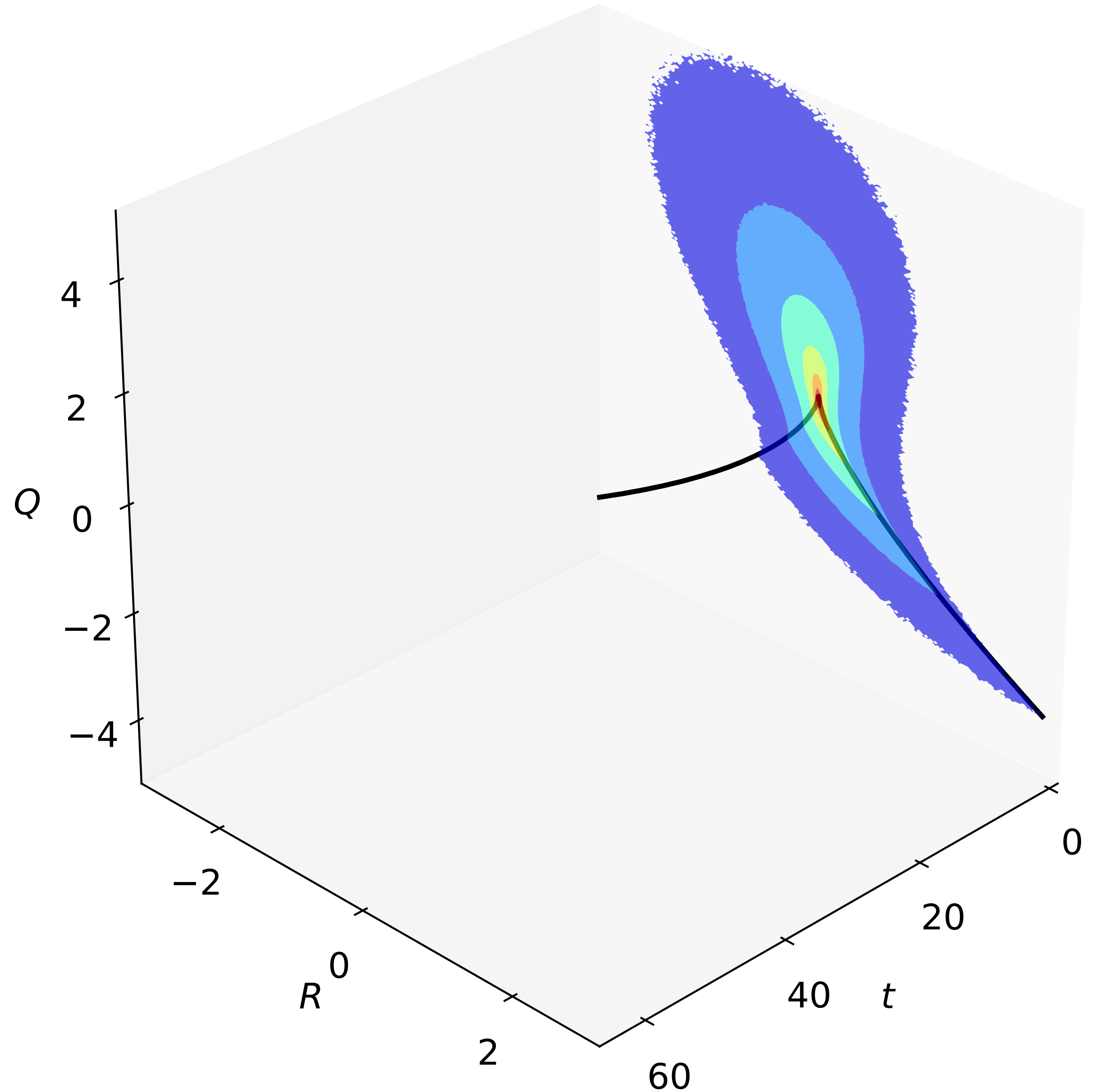
$$T_{ijpq} = -T_{pqij}$$

T(A;ψ): "Gauge" terms
anti-symmetric

*f(**A**)* imposed,
T(A) to be learned

Bridging single- and multi-time statistics

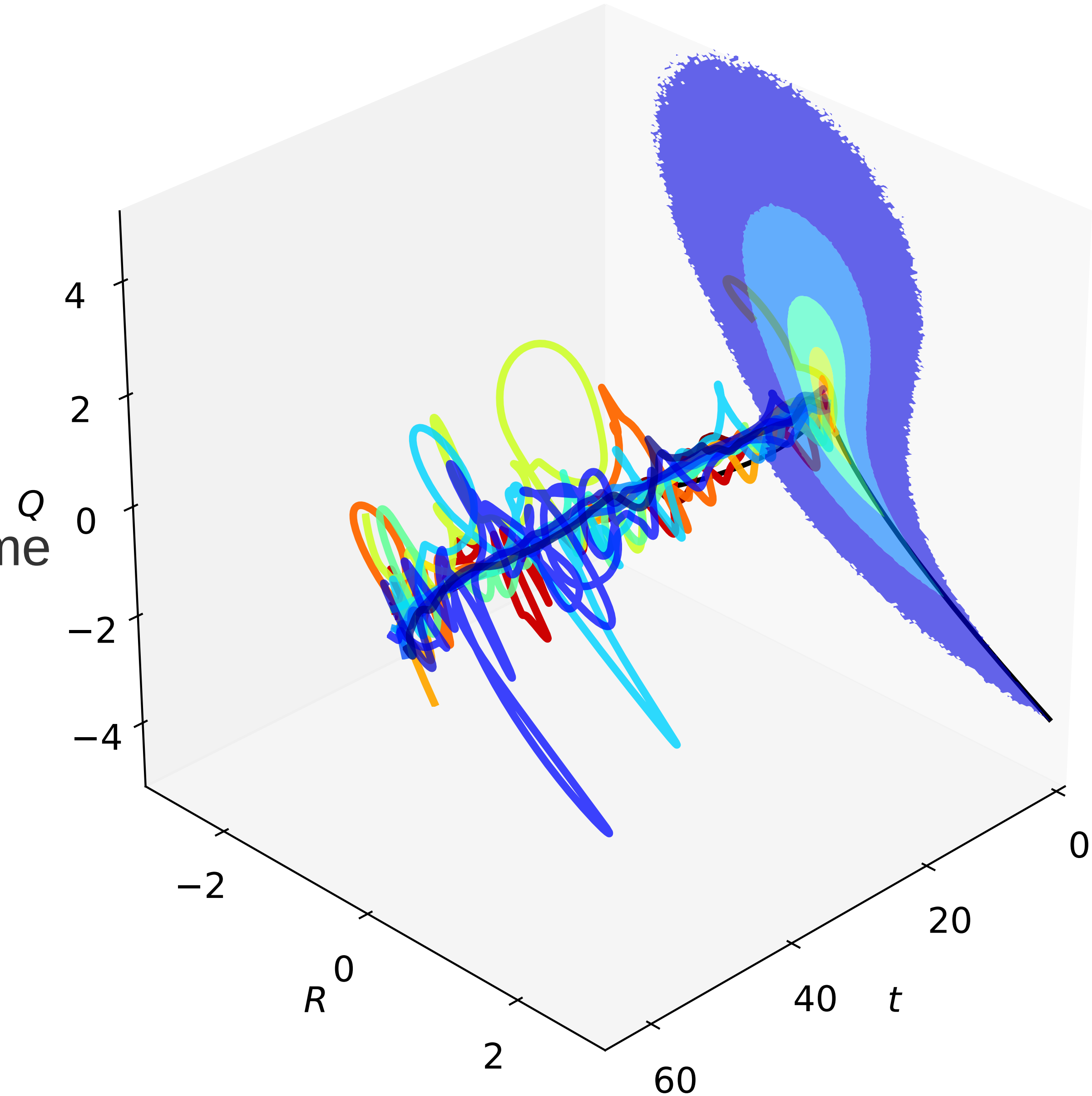
- Learn single-time PDF
- Construct system featuring that steady-state PDF



Bridging single- and multi-time statistics

- Learn single-time PDF
- Construct system featuring that steady-state PDF
- Optimize trajectories: time correlations, conditional dynamics, (GAN, diffusive models, etc.)

[Li, Biferale, Bonaccorso, Scarpolini and Buzzicotti, arXiv physics.flu-dyn, 2023]



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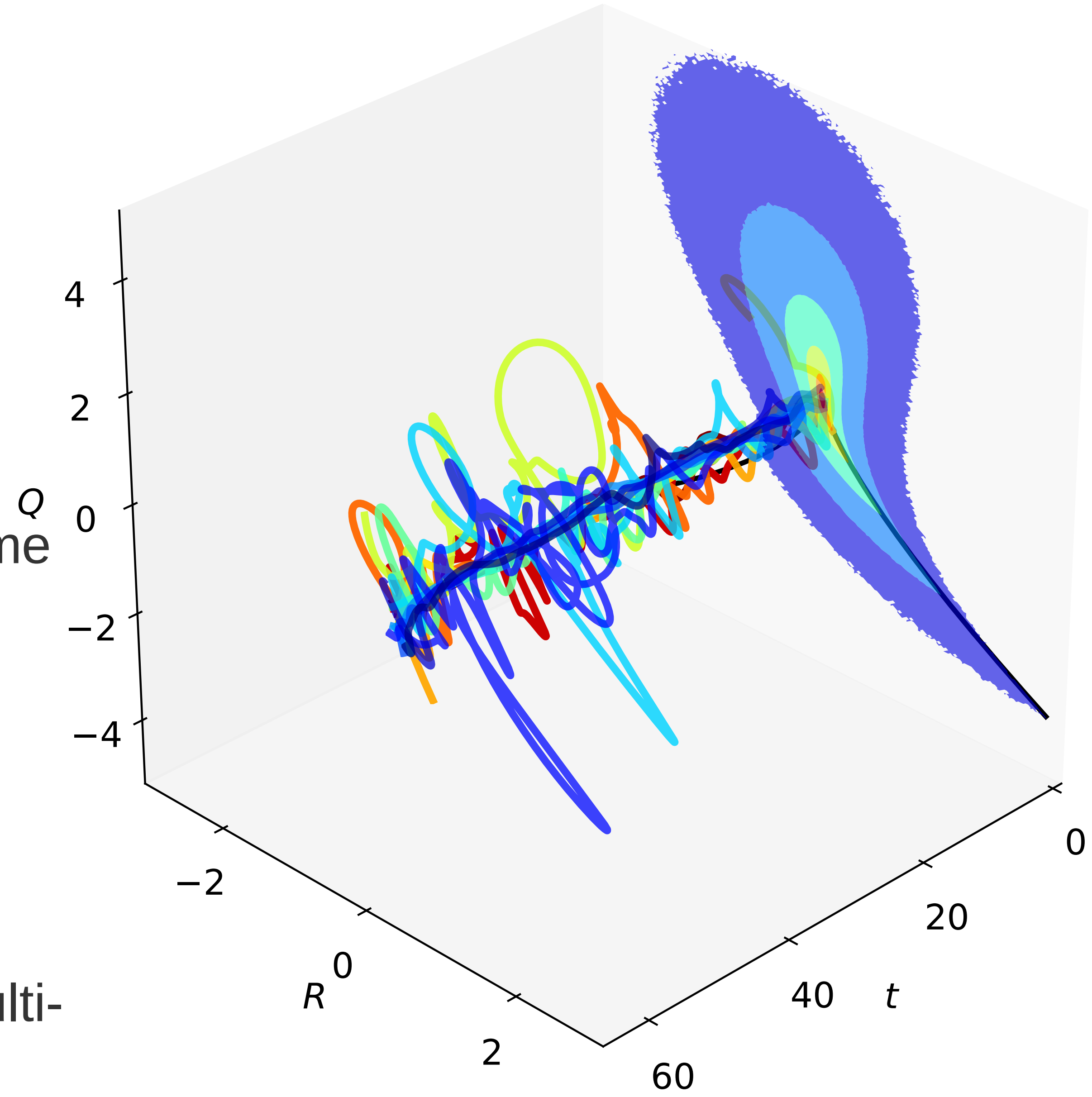
  [Li, Biferale, Bonaccorso, Scarpolini and Buzzicotti, arXiv physics.flu-dyn, 2023]

- *Independent s*ingle- and multi-time optimizations



Bridging single- and multi-time statistics

- Lagrangian realizations of the gradient from DNS ***A***(t)

- Numerically integrate model realizations

$$\mathrm{d}_t \boldsymbol{\mathcal{A}} = \boldsymbol{N}(\boldsymbol{\mathcal{A}})$$

Neural Net.*(**A**;ψ)*

Bridging single- and multi-time statistics

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Neural Net.***(A;ψ)***

- with specific drift

$$N_{ij}(\boldsymbol{\mathcal{A}};\psi) = \frac{\partial T_{ijpq}}{\partial \mathcal{A}_{pq}} + T_{ijpq}\frac{\partial \log f}{\partial \mathcal{A}_{pq}}$$

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- with specific drift

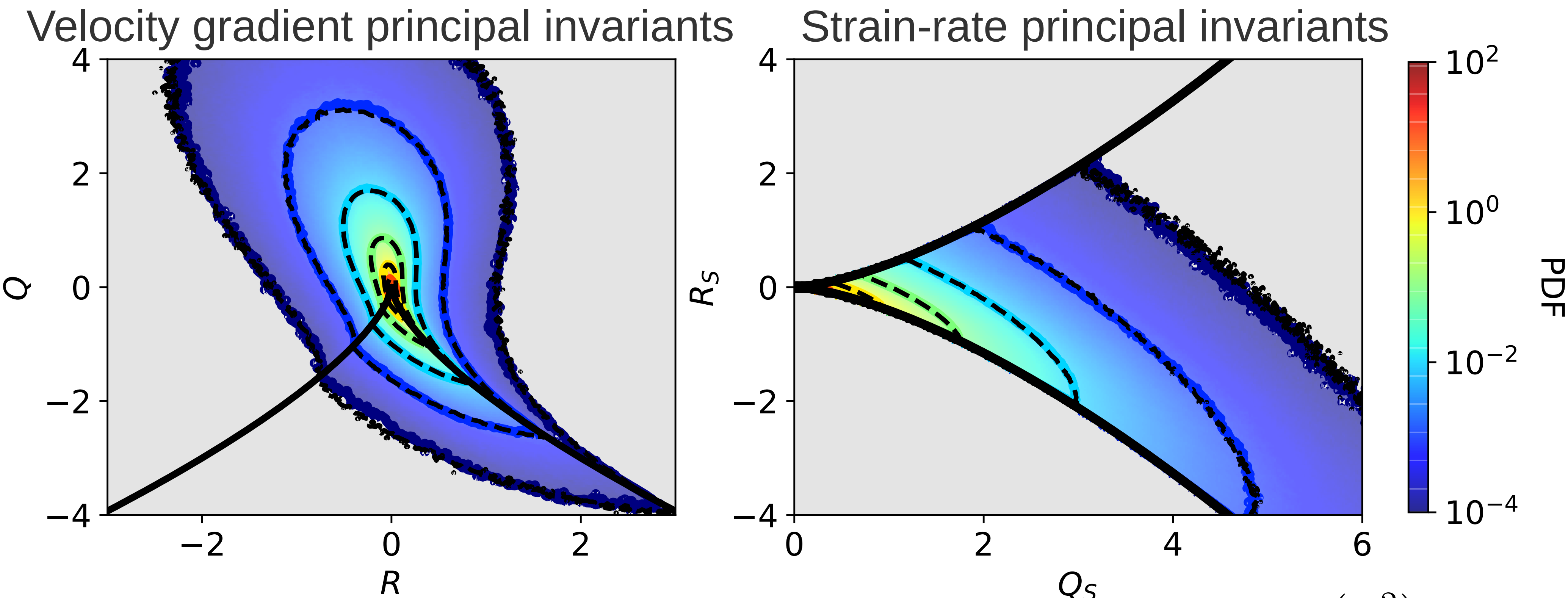
$$N_{ij}(\boldsymbol{\mathcal{A}};\psi) = \frac{\partial T_{ijpq}}{\partial \mathcal{A}_{pq}} + T_{ijpq}\frac{\partial \log f}{\partial \mathcal{A}_{pq}}$$

- Optimize e.g., time correlations and conditional derivatives

DNS Model

$$\min_{\psi}\left[\left\|\langle A_{ij}(t_0)A_{pq}(t)\rangle_0 - \langle \mathcal{A}_{ij}(t_0;\psi)\mathcal{A}_{pq}(t;\psi)\rangle_0\right\|^2 + \left\langle |\boldsymbol{B}_i : (\mathrm{d}_t\boldsymbol{A} - \boldsymbol{N}(\boldsymbol{A}))|^2\right\rangle\right]$$

Single-time statistics: principal invariants PDF



$$Q = -\mathrm{Tr}\left(\boldsymbol{A}^2\right)/2$$

$$R = -\mathrm{Tr}\left(\boldsymbol{A}^3\right)/3$$

DNS

Model

$$Q_S = \mathrm{Tr}\left(\boldsymbol{S}^2\right)$$

$$R_S = \mathrm{Tr}\left(\boldsymbol{S}^3\right)$$

Skewness,
vorticity intermittency,
Vieillefosse

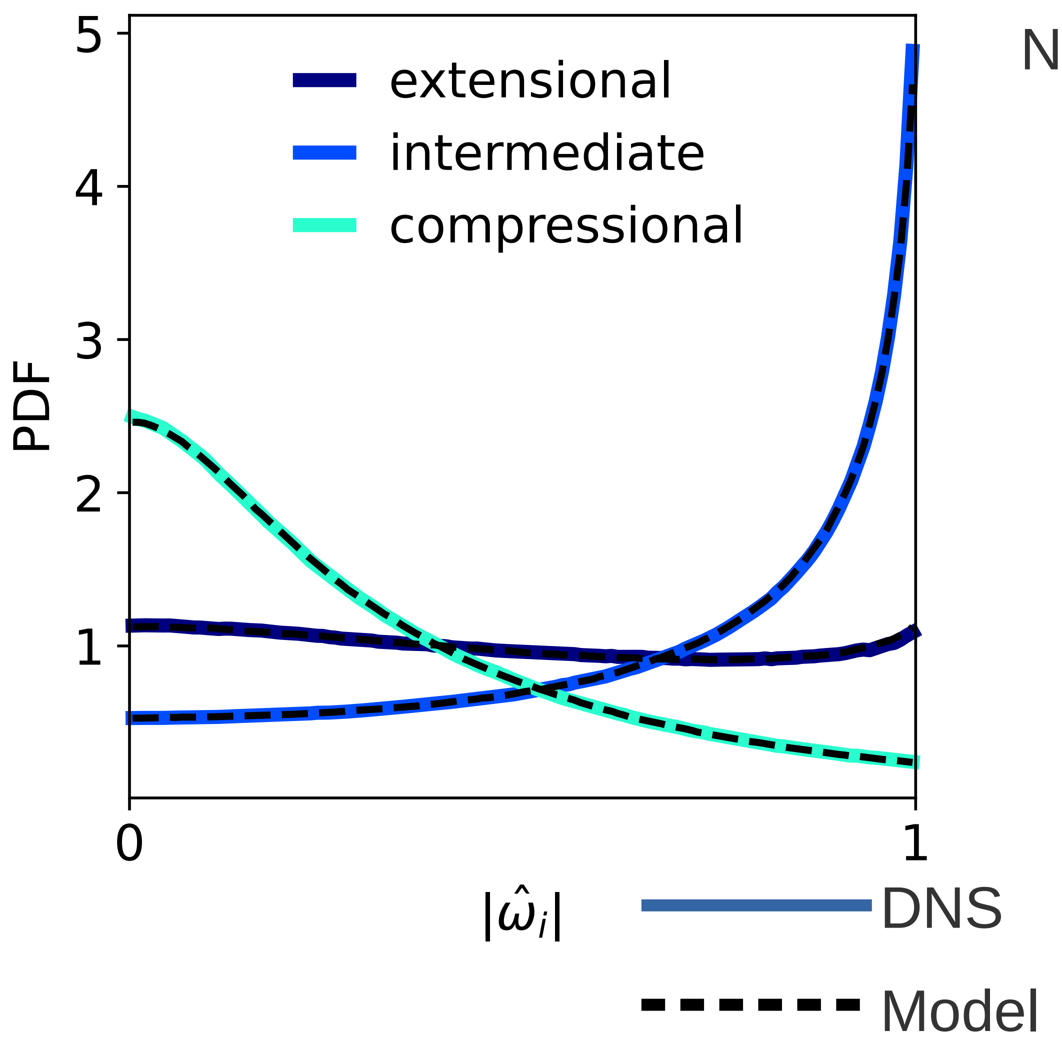
Dissipation rate,
energy transfer, etc.

Kolmogorov units

$$\tau_\eta = \frac{1}{\sqrt{2\left\langle Q_S \right\rangle}} = 1$$

Single-time statistics: vorticity principal components PDF

Strain rate–vorticity alignments



Normalized vorticity components in the strain-rate eigenframe

$$\boldsymbol{S} = \frac{1}{2}\left(\boldsymbol{A} + \boldsymbol{A}^\top\right)$$

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$$

$$\hat{\omega}_i = \boldsymbol{\omega} \cdot \boldsymbol{v}_i / \|\boldsymbol{\omega}\|$$

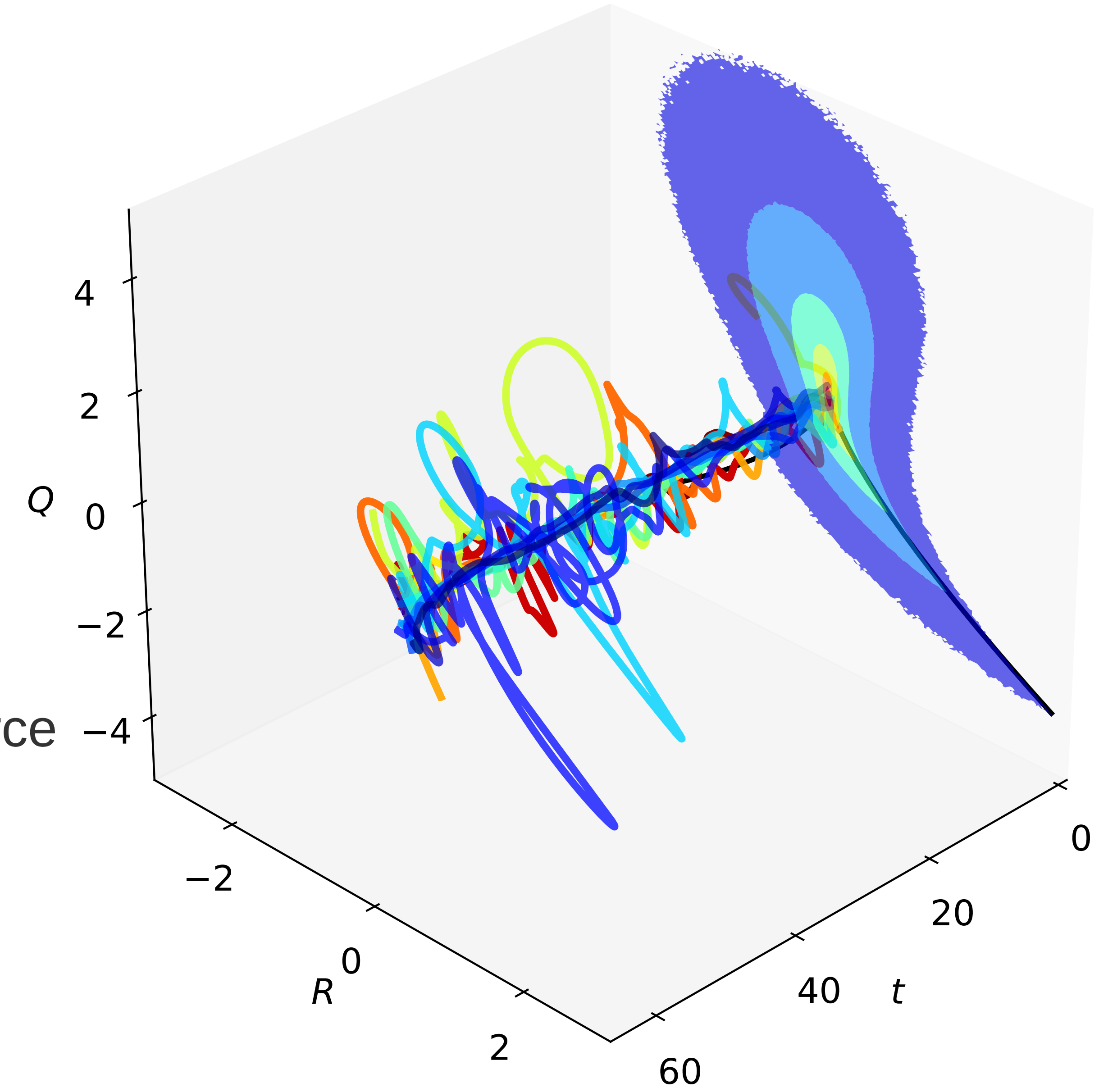
Strain-rate eigenvectors associated with ordered eigenvalues

Now two-time statistics

- Learn single-time PDF

- Construct system featuring that steady-state PDF

- Optimize trajectories: enforce time correlations



Time correlations and sample realizations

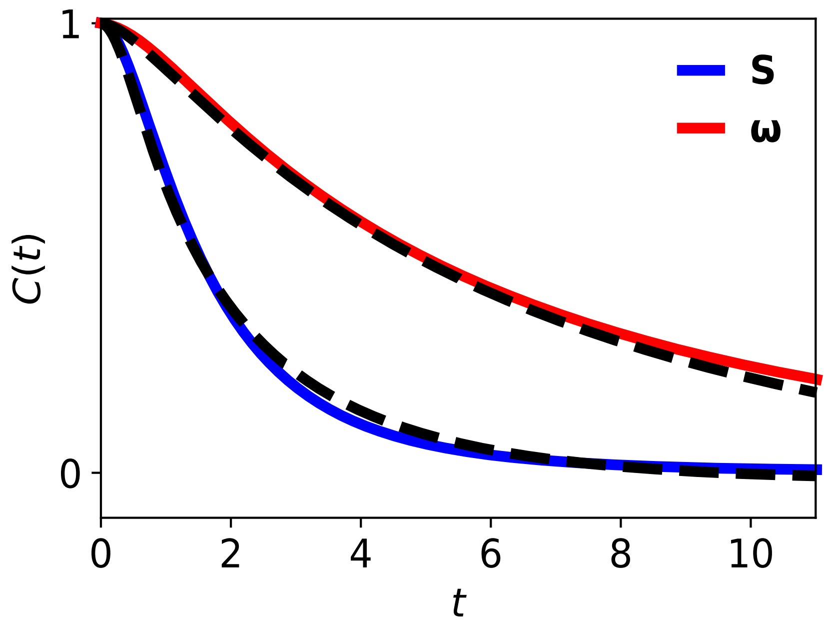
Normalized correlations

Vorticity:

$$C_{\boldsymbol{\omega}}(t) = \frac{\langle \omega_i(0)\omega_i(t)\rangle}{\langle \omega_i(0)\omega_i(0)\rangle}$$

Strain rate:

$$C_{\mathbf{S}}(t) = \frac{\langle S_{ij}(0)S_{ij}(t)\rangle}{\langle S_{ij}(0)S_{ij}(0)\rangle}$$



Time correlations and sample realizations

Normalized correlations

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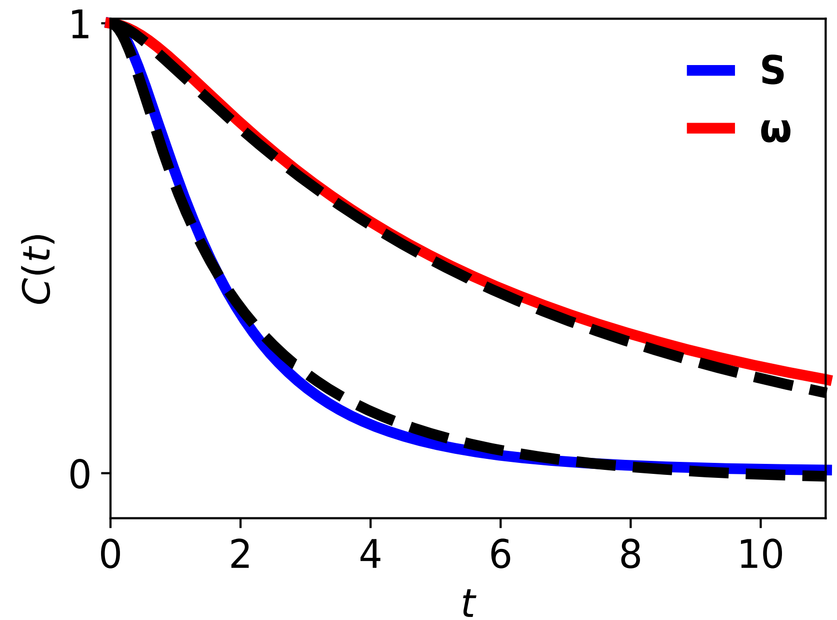
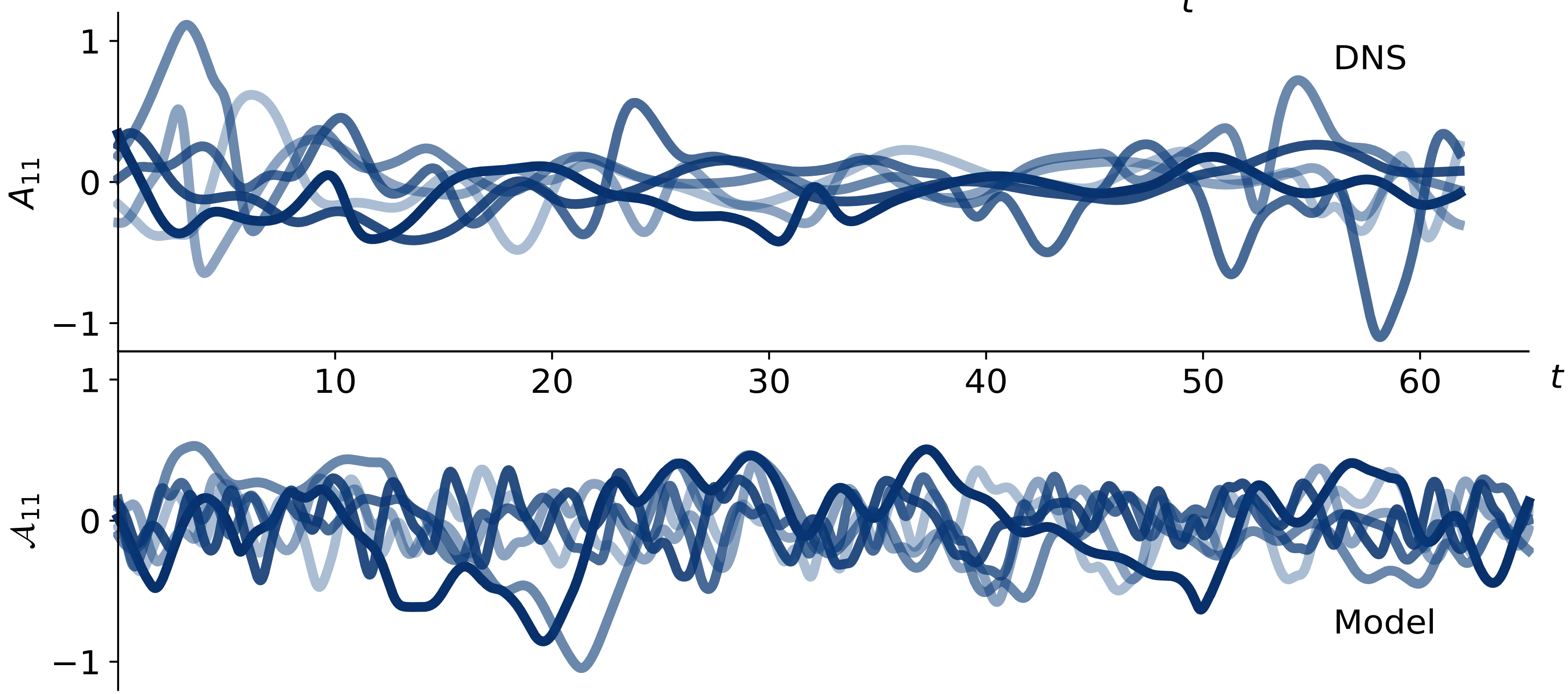
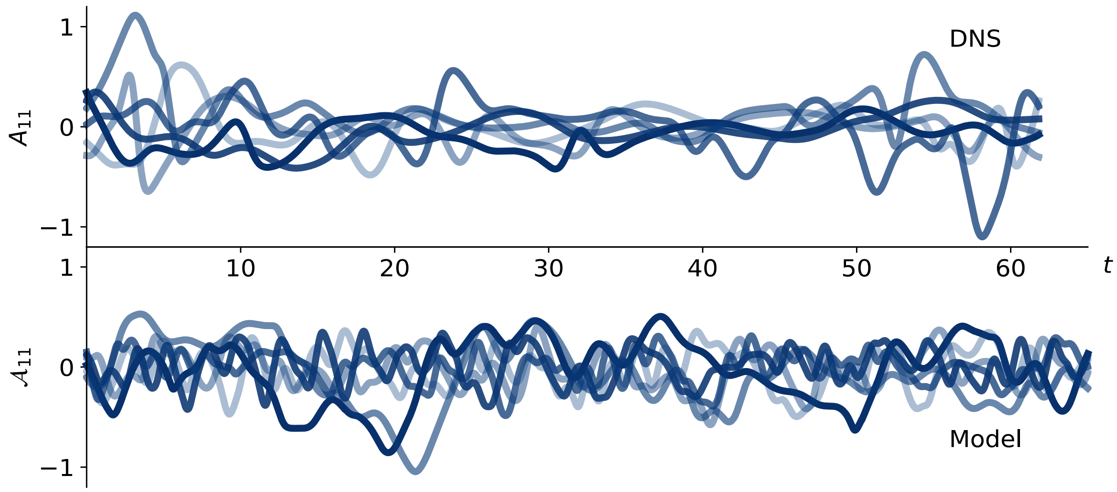




Rely on chaos for a non-trivial model

Chaotic dynamical system:
[S. H. Strogatz, Nonlinear Dynamics and Chaos, (2000)]

- Deterministic, aperiodic



Rely on chaos for a non-trivial model

Chaotic dynamical system:

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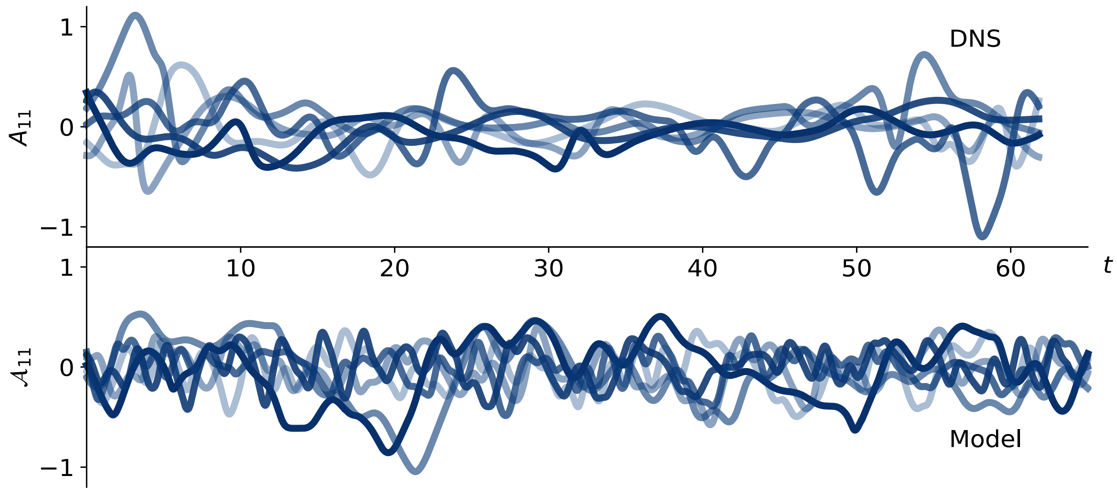
- Deterministic, aperiodic
- Positive Lyapunov exponent



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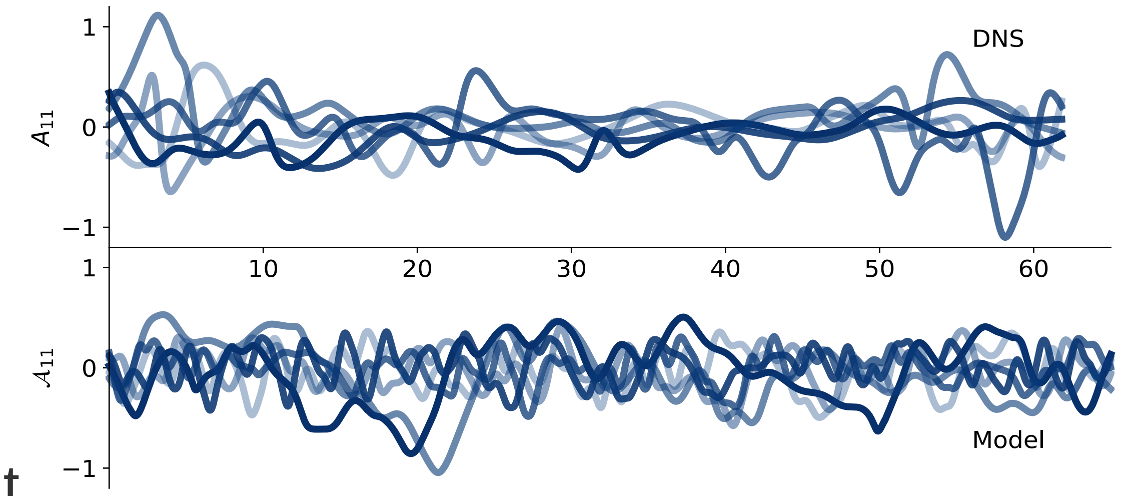
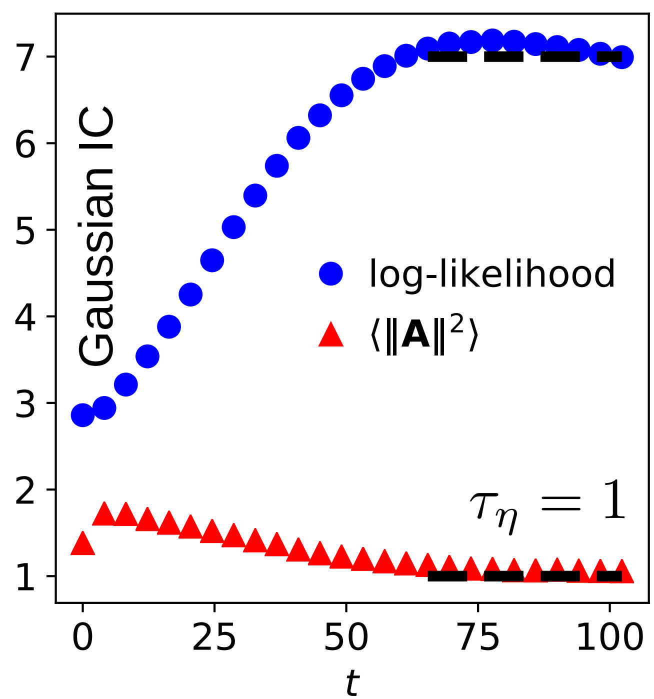
- Positive Lyapunov exponent
- Converge from Gaussian to ~turbulent ensemble





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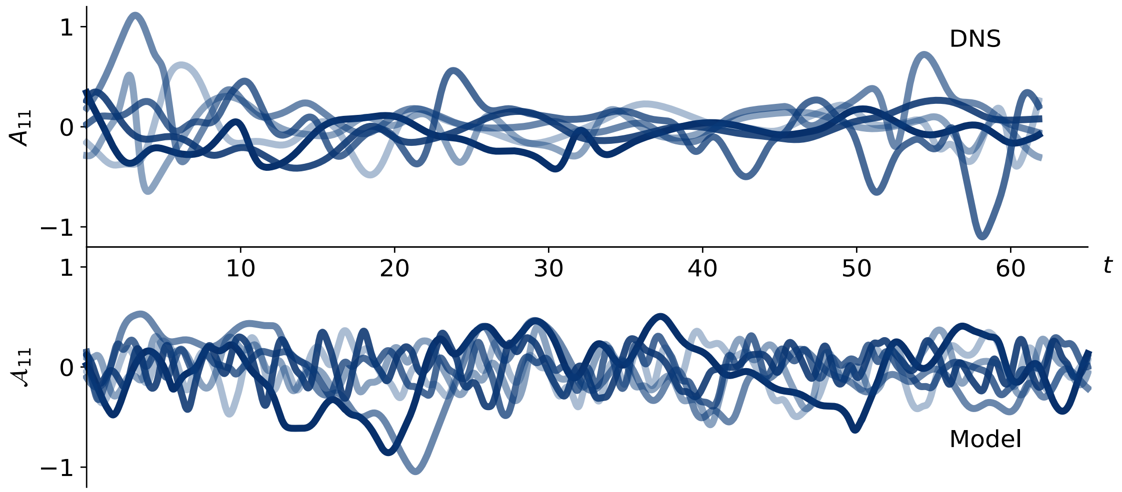
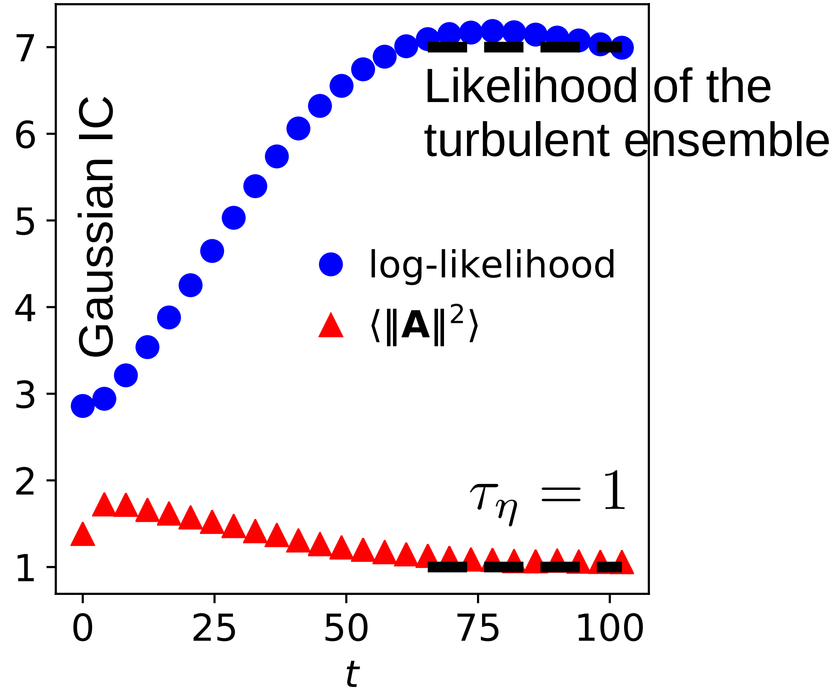
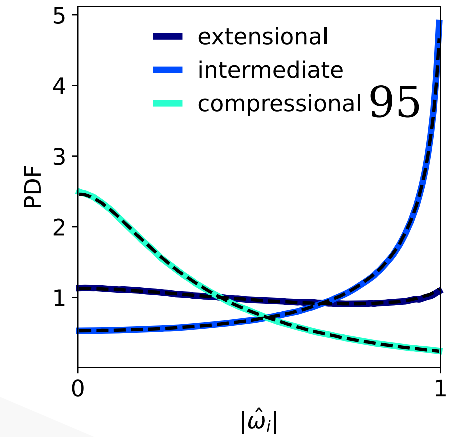
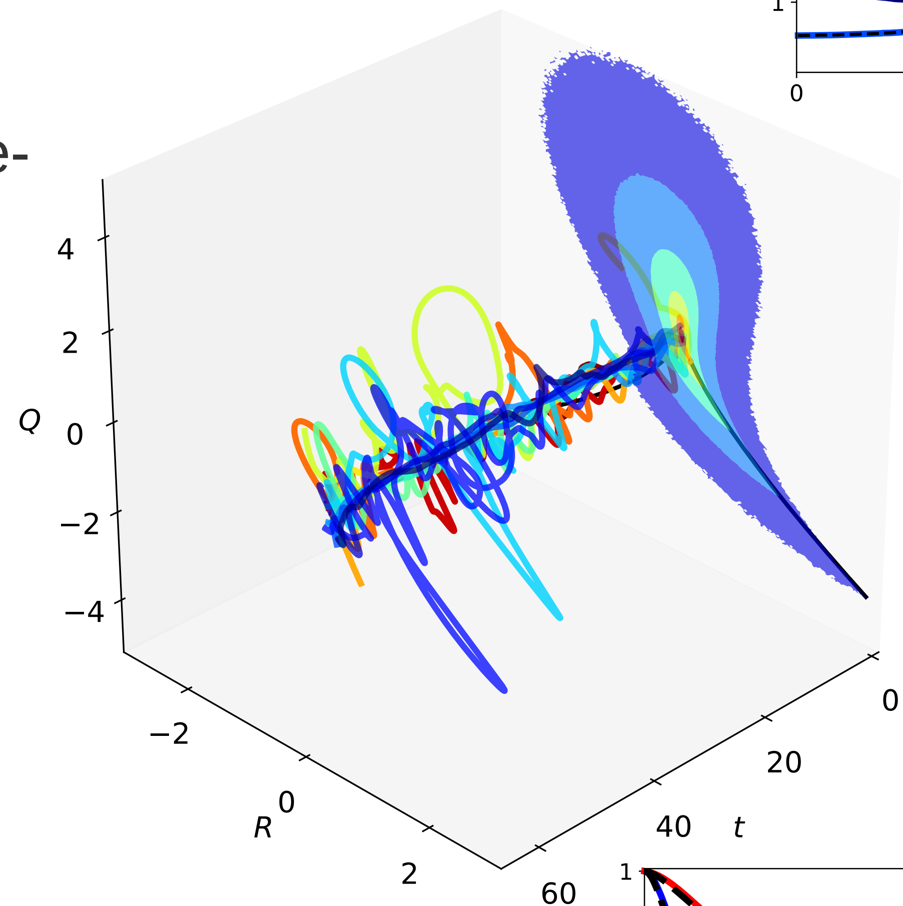


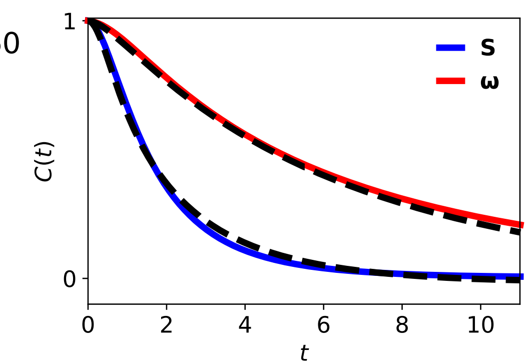


Conclusions: Tailor-designed model for the turbulent velocity gradients



- Normalizing flow to learn single-time PDF
- Single-time PDF ~exact by construction
- Independent multi-time optimization for trajectories
- Deterministic, chaotic system





Something recurrent:
The kinematics of the velocity gradients

- Homogeneity and incompressibility

$$\langle \phi \rangle = \boldsymbol{\nabla} \cdot \langle F \rangle = 0$$

- Quantity = divergence of some field
- Homogeneity/no-flux implies zero average

Carbone and Wilczek, “Only two Betchov homogeneity constraints exist for isotropic turbulence”, *JFM* **948**, (2022)

Something recurrent:
The kinematics of the velocity gradients

- Homogeneity and incompressibility

$$\langle \phi \rangle = \nabla \cdot \langle F \rangle = 0$$

- Quantity = divergence of some field
- Homogeneity/no-flux implies zero average

$$\mathrm{Tr}\left(\boldsymbol{A}^2\right) = \nabla_j u_i \nabla_i u_j = \nabla_i \left(u_j \nabla_j u_i\right)$$

$$\mathrm{Tr}\left(\boldsymbol{A}^3\right) = \nabla_j u_i \nabla_k u_j \nabla_i u_k = \nabla_i \left(u_k \nabla_j u_i \nabla_k u_j - \frac{1}{2} u_i \nabla_k u_j \nabla_j u_k\right)$$

- Easy to show the Betchov relations..
- ..but how to find all possible homogeneity relations?

Carbone and Wilczek, “Only two Betchov homogeneity constraints exist for isotropic turbulence”, *JFM* **948**, (2022)

Kinematics of the velocity gradients

- Write the most general **F**

$$\langle \phi \rangle = \nabla \cdot \langle F \rangle = 0$$

- Impose divergence function of only **A**

$$\phi(\boldsymbol{A}) = \frac{\partial F_i}{\partial u_p}(\boldsymbol{u}, \boldsymbol{A})\, A_{pi}$$

$$\frac{\partial F_i}{\partial A_{pq}}(\boldsymbol{u}, \boldsymbol{A})\, \nabla_i A_{pq} = 0$$

- Homogeneity constraints: solutions of linear PDE
- All solutions $\Longleftrightarrow$ All constraints

Carbone and Wilczek, "Only two Betchov homogeneity constraints exist for isotropic turbulence", *JFM* **948**, (2022)

Kinematics of the velocity gradients

- Some generalization

$$\langle \phi \rangle = \nabla \cdot \langle F \rangle = 0$$

$$\psi\left(\boldsymbol{A}, \boldsymbol{\nabla q}\right) = \boldsymbol{\nabla} \cdot \left[\bar{c}_1 \boldsymbol{q} + \bar{c}_2 \boldsymbol{A q} + \bar{c}_2 \left(\boldsymbol{A}^2 - \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{A}^2\right)\boldsymbol{I}\right)\boldsymbol{q}\right]$$

- Several relations on pressure, Laplacian, vorticity, etc.

$$\left\langle A_{ij} \nabla^2 A_{ji} \right\rangle = 0$$

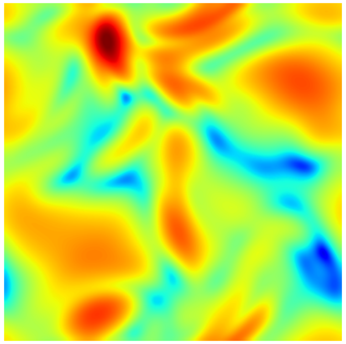
$$\left\langle A_{ij}^2 \nabla^2 A_{ji} \right\rangle = 0$$

$$\left\langle A_{ij}^2 \nabla_i \nabla_j P \right\rangle = -\frac{\rho}{2}\left\langle \left(A_{ij} A_{ji}\right)^2 \right\rangle$$

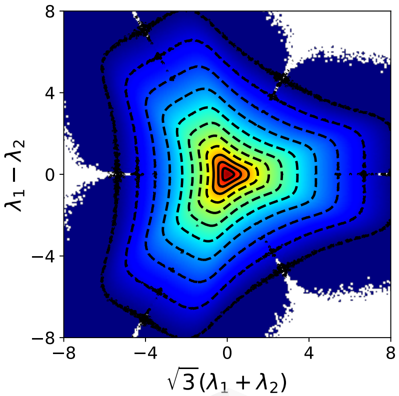
$$\left\langle A_{ij} \nabla_i \omega_j \right\rangle = 0$$

$$\left\langle A_{ij}^2 \nabla_i \omega_j \right\rangle = 0$$

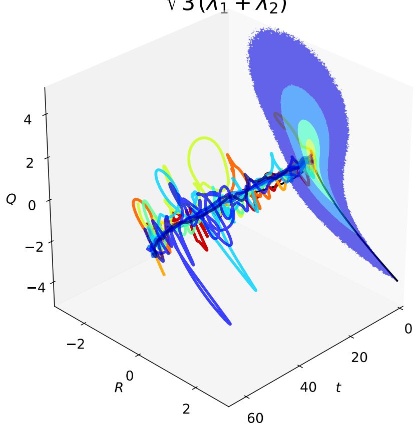
Eyink, JFM **549**, (2006)
Capocci, Johnson, Oughton, Biferale, Linkmann, JFM **963**, (2023)



Velocity gradients at low Reynolds numbers



Strain rate at high Reynolds numbers



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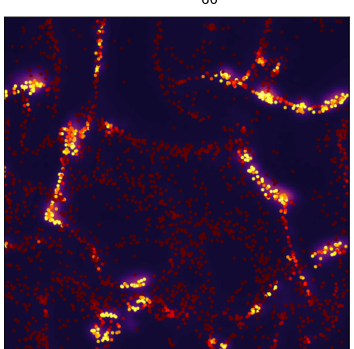

Some applications: Quantifying energy cascade

Acknowledgments: Prof. Andy Bragg

Coarse-grained velocity gradients and velocity increments

- Coarse-grained gradient at varying scale *r*
- Tilde: filtering at scale $\ell(r)$
- Incompressibility issue! $\boldsymbol{\partial_r} \cdot \Delta \widetilde{\boldsymbol{u}} \neq 0$

$$\boxed{\Delta \widetilde{\boldsymbol{u}} \simeq \widetilde{\boldsymbol{A}} \cdot \boldsymbol{r}}$$

Carbone and Bragg, “Is vortex stretching the main cause of the turbulent energy cascade?”, JFM 883, (2020)

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$$\boxed{\Delta\widetilde{\boldsymbol{u}} \simeq \widetilde{\boldsymbol{A}} \cdot \boldsymbol{r}}$$

$$\Delta^*\widetilde{\boldsymbol{u}} = \boldsymbol{\partial_r} \times \widetilde{\boldsymbol{V}^*}(\boldsymbol{x}, \ell(r), t)$$

$$\Delta^*\widetilde{\boldsymbol{u}}\big|_\ell = \Delta\widetilde{\boldsymbol{u}}$$

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$$\Delta^* \widetilde{\boldsymbol{u}}\big|_\ell = \Delta \widetilde{\boldsymbol{u}}$$

- Correction on the transverse increment only

$$\Delta^* \widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{A}} \cdot \boldsymbol{r} + \frac{\boldsymbol{r}}{r} \times \left[\frac{1}{2} (\boldsymbol{rr} : \boldsymbol{\nabla\nabla}) \partial_r \widetilde{\boldsymbol{V}}(\boldsymbol{x}, \ell(r), t) + \partial_r \widetilde{\boldsymbol{C}}(\boldsymbol{x}, r, t) \right] + \boldsymbol{h}$$

$$\left\langle \Delta^* \widetilde{u}_\perp^2 \Delta^* \widetilde{u}_\parallel \right\rangle = \frac{1}{6} \partial_r \left\langle r \Delta^* \widetilde{u}_\parallel^3 \right\rangle$$

Carbone and Bragg, “Is vortex stretching the main cause of the turbulent energy cascade?”, JFM 883, (2020)

Coarse-grained velocity gradients and velocity increments

- Coarse-grained gradient at varying scale *r*
- Solve incompressibility issue

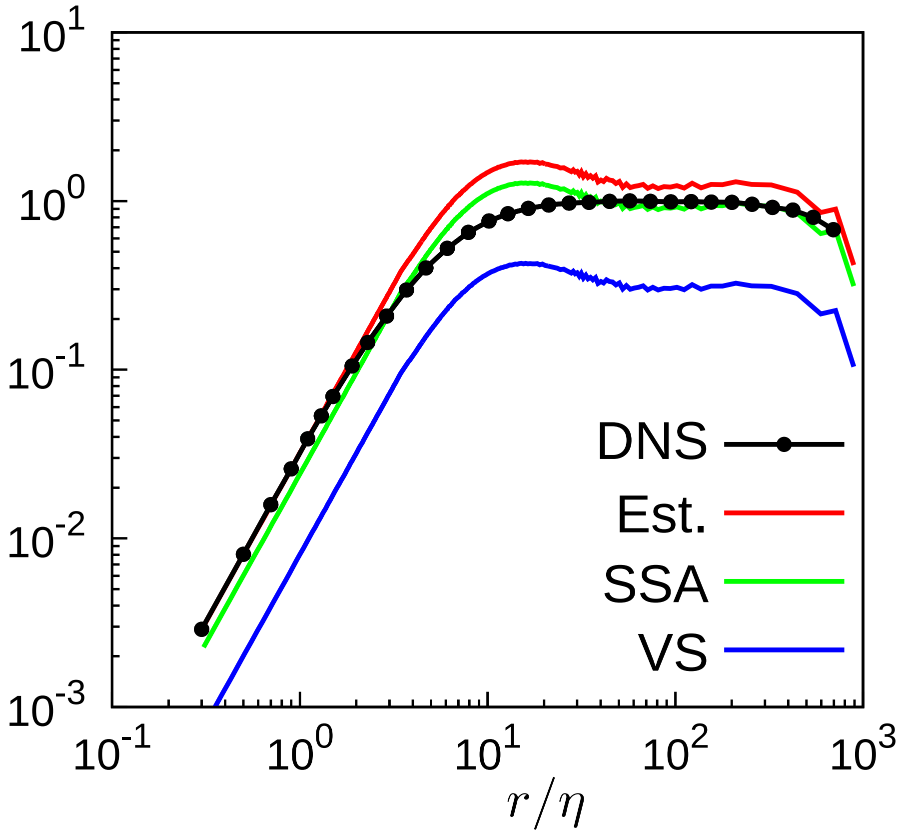
$$\boxed{\Delta\widetilde{\boldsymbol{u}} \simeq \widetilde{\boldsymbol{A}} \cdot \boldsymbol{r}}$$

$$\Delta^*\widetilde{\boldsymbol{u}} = \boldsymbol{\partial}_{\boldsymbol{r}} \times \left(2\widetilde{\boldsymbol{V}}(\boldsymbol{x}+\boldsymbol{r}/2,t) + 2\widetilde{\boldsymbol{V}}(\boldsymbol{x}-\boldsymbol{r}/2,t) + \widetilde{\boldsymbol{B}}(\boldsymbol{x},t)\right)$$

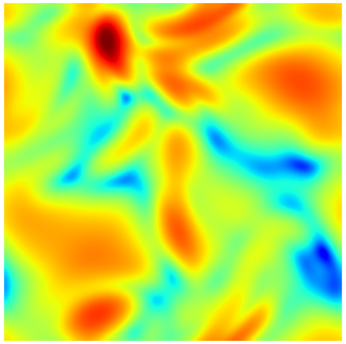
- Energy transfer across the scales

$$\boldsymbol{T} = \|\Delta^*\widetilde{\boldsymbol{u}}\|^2\Delta^*\widetilde{\boldsymbol{u}}$$

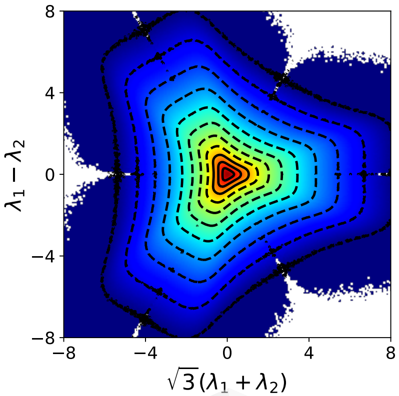
$$\boldsymbol{\partial}_{\boldsymbol{r}} \cdot \boldsymbol{T} = \mathcal{L}_r\left[\underbrace{\left((\widetilde{\boldsymbol{S}}\cdot\widetilde{\boldsymbol{S}}):\widetilde{\boldsymbol{S}}\right)}_{\text{SSA}} - \frac{1}{4}\underbrace{\left(\widetilde{(\boldsymbol{\omega}\cdot\widetilde{\boldsymbol{S}})}\cdot\widetilde{\boldsymbol{\omega}}\right)}_{\text{VS}}\right]$$



Carbone and Bragg, "Is vortex stretching the main cause of the turbulent energy cascade?", *JFM* **883**, (2020)



Velocity gradients at low Reynolds numbers



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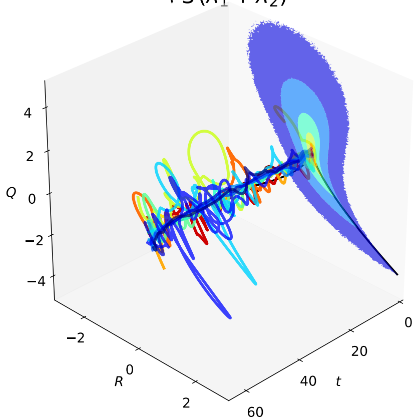

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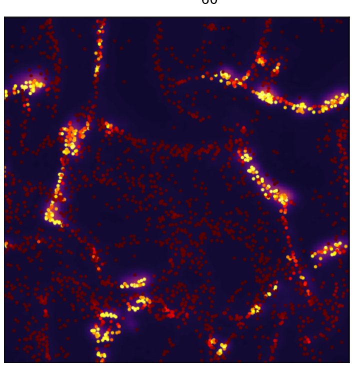

Some applications: Iron particle combustion

Acknowledgments: Ing. Gabriel Thäter, Prof. Bettina Frohnapfel, Prof. Oliver T. Stein

Iron particle combustion in turbulence

- Physical model: variable-density Navier-Stokes

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = C^\rho$$

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + F_i^u + C_i^u$$

$$\frac{\partial H}{\partial t} + \frac{\partial (u_i H)}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial T}{\partial x_i}\right) + \frac{\mathrm{d}p_0}{\mathrm{d}t} + F^H + C^H$$

- Low-Mach approximation

$$P(\boldsymbol{x}, t) = p_0(t) + \pi(\boldsymbol{x}, t)$$

$$\frac{\partial (\rho u_i)}{\partial x_i} = \mathcal{F}[\rho, \boldsymbol{u}, T, m_p, T_p].$$

Iron particle combustion in turbulence

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$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = \boxed{C^\rho}$$

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$$\frac{\partial(\rho u_i)}{\partial x_i} = \mathcal{F}[\rho, \boldsymbol{u}, T, m_p, T_p].$$

- Coupling via Non-Uniform FFT

$$\boldsymbol{C}^u(\boldsymbol{x}, t) = \sum_{p=1}^{N_P} M_{3,p}(t)\left(\boldsymbol{u}(\boldsymbol{x}_p, t) - \boldsymbol{v}_p\right)\delta\left(\boldsymbol{x} - \boldsymbol{x}_p\right)$$

Carbone, Iovieno, Bragg, “Multiscale fluid--particle thermal interaction in isotropic turbulence”, *JFM* **881**, (2019)

Iron particle combustion in turbulence

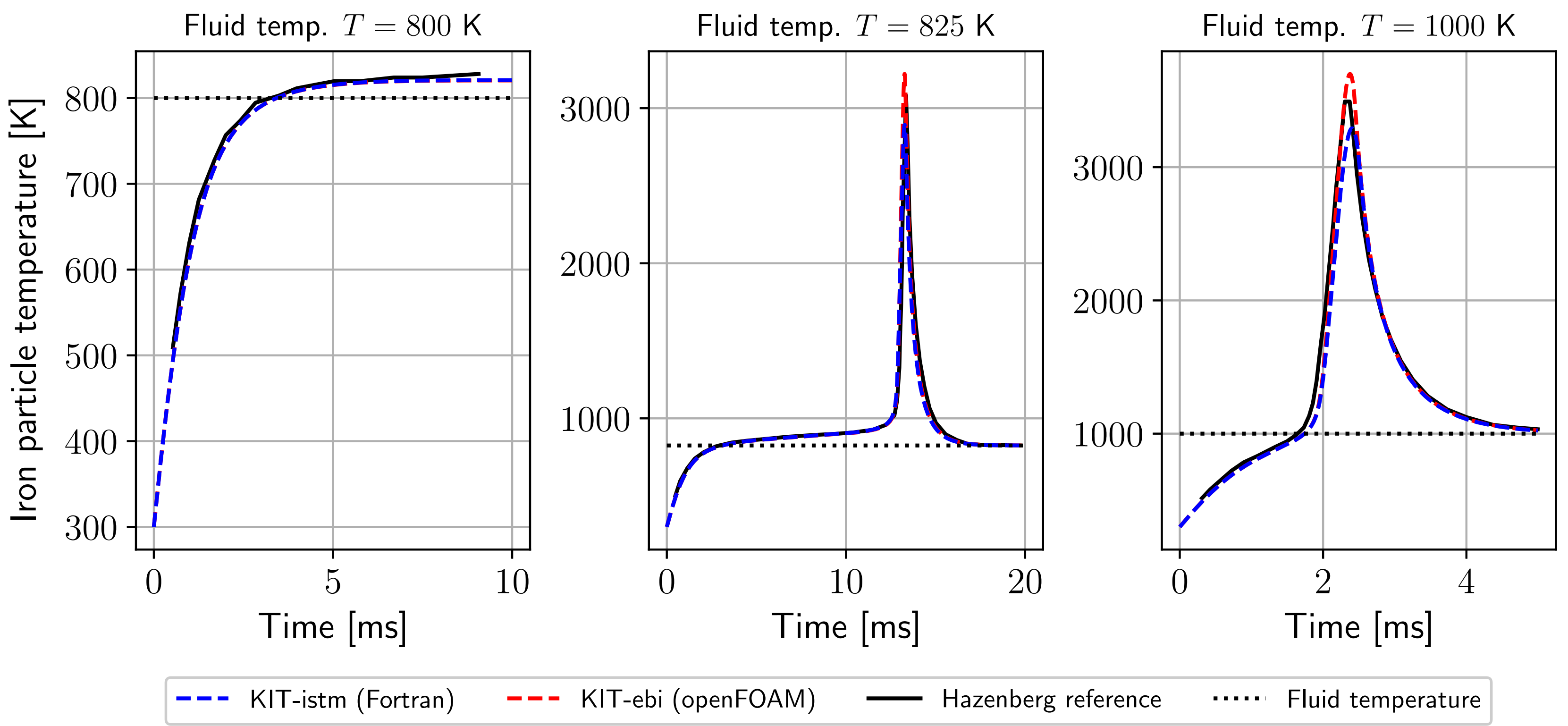
- Physical model: Reacting iron particles

$$\frac{\mathrm{d}m_p}{\mathrm{d}t} = M_{1,p}\rho(\boldsymbol{x}_p, t) = M_{2,p}\frac{\mathrm{d}m_{p,\mathrm{Fe}}}{\mathrm{d}t}$$

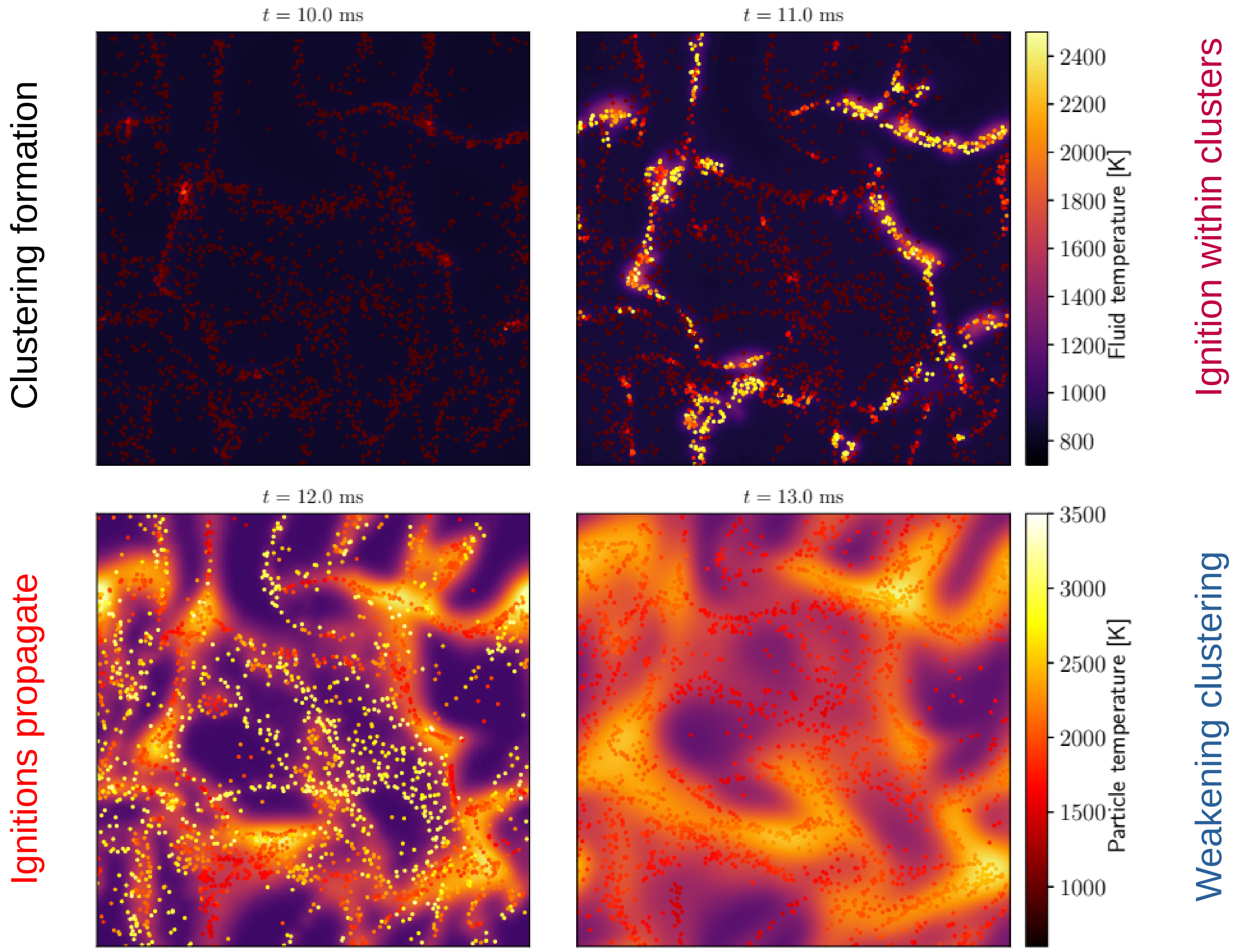
$$\frac{\mathrm{d}^2\boldsymbol{x}_p}{\mathrm{d}t^2} = \frac{\mathrm{d}\boldsymbol{v}_p}{\mathrm{d}t} = M_{3,p}\left(\boldsymbol{u}(\boldsymbol{x}_p, t) - \boldsymbol{v}_p\right)$$

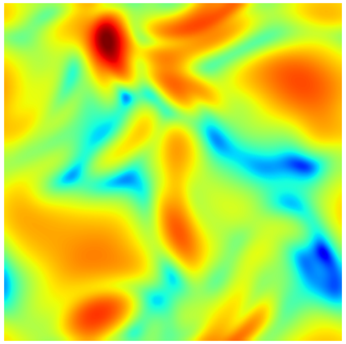
$$\frac{\mathrm{d}H_p}{\mathrm{d}t} = M_{4,p}(T(\boldsymbol{x}_p, t) - T_p) + M_{5,p}\frac{\mathrm{d}m_p}{\mathrm{d}t}$$

Hazenberg, van Oijen, Proc. Combust. Inst. **38**(3), (2021)



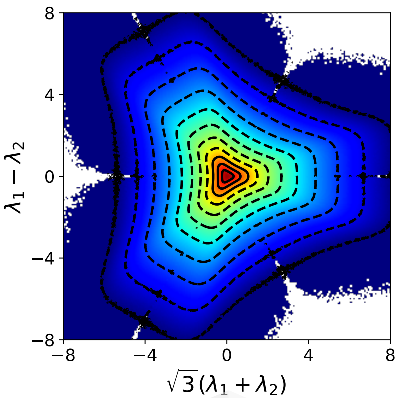
Gather, react, eject

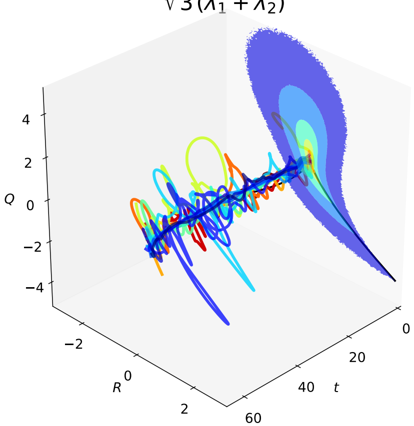


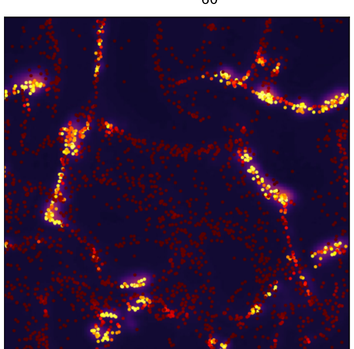


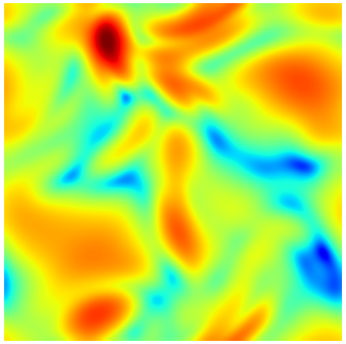
Velocity gradients at low Reynolds numbers

Analytic insight, onset of skewness, alignments and intermittency



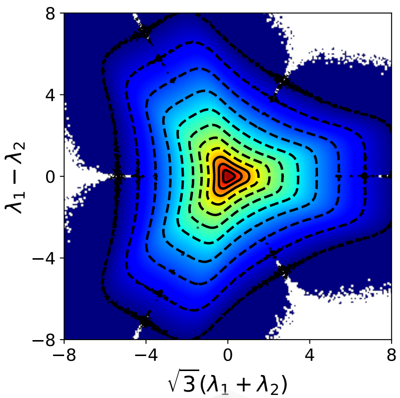






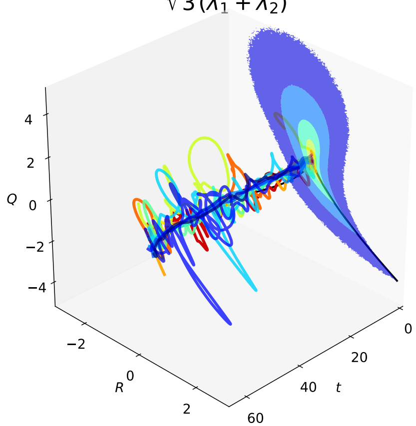
Velocity gradients at low Reynolds numbers

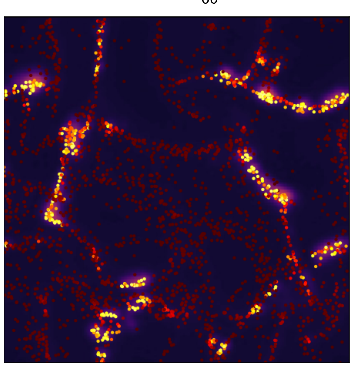
Analytic insight, onset of skewness, alignments and intermittency

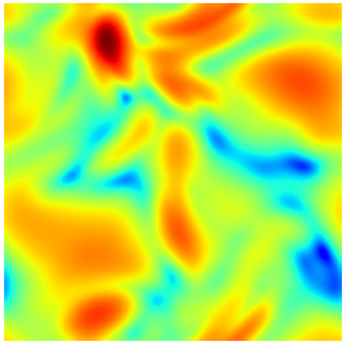


Strain rate at high Reynolds numbers

Analytically parameterized the strain-rate PDF, sampled PDF via tailor-made model







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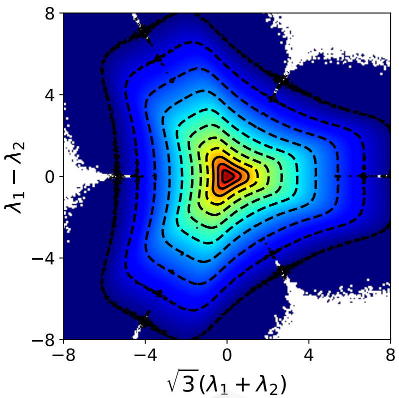

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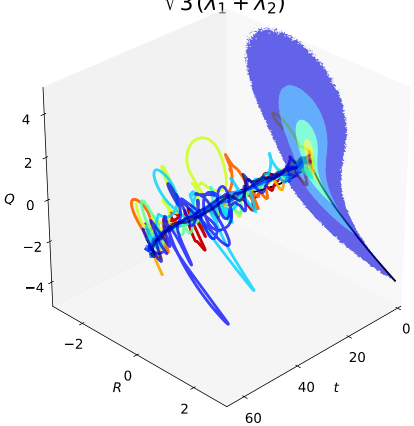
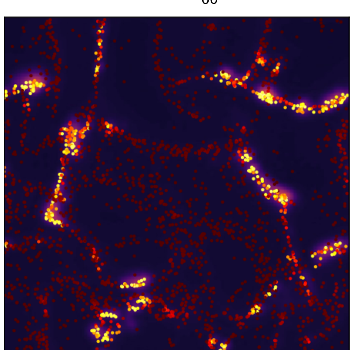
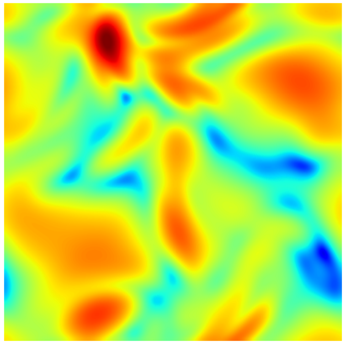


Velocity gradients at high Reynolds numbers

Normalizing flow to learn the velocity gradient PDF, deterministic, chaotic model for the small scales





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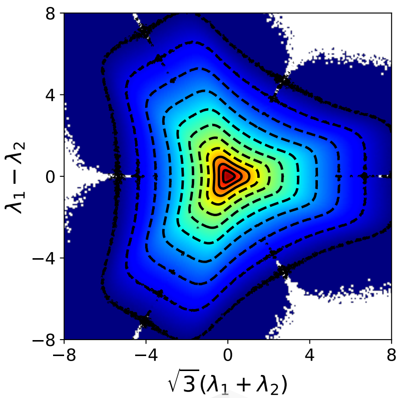

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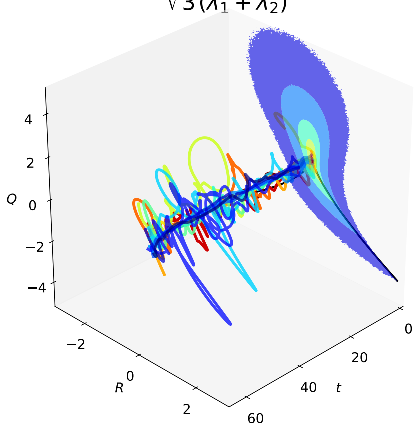

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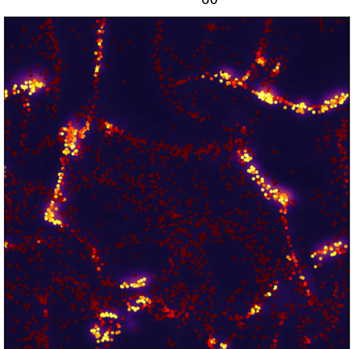

Some applications

Quantify average energy cascade,
Turbulence interacting with iron particle combustion